Pier Paolo Delsanto

Editor

Universality of Nonclassical Nonlinearity

Applications to Non-Destructive Evaluations and Ultrasonics

Pier Paolo Delsanto
Dip. Fisica
24 Corso Duca degli Abruzzi
Torino 10129
Italy

Universality in Nonlinear Structural Dynamics

Alberto Carpinteri and Nicola Pugno[1]

Department of Structural Engineering and Geotechnics, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129, Torino, Italy.
[1]To whom correspondence should be addressed: nicola.pugno@polito.it

Abstract
The aim of this chapter is the investigation of the universal behavior of nonlinear vibrating structures. The approach is formulated in terms of the black box interaction formalism, only recently introduced (see Chapter 1), but including subharmonics and multidegrees of freedom. As a prototype of the nonlinear box, we focus attention on a multicracked cantilever beam. The cause of the vibration (i.e., the input in the box) is represented by a harmonic force excitation; the effect (i.e., the output), by the tip displacement. Universality corresponds to zero-, high-, and subharmonic generations, describing complex phenomena such as period doublings and transition towards deterministic chaos. Applications to damage detection and structural monitoring seem to be promising.

Keywords: Chaos, complexity, cracks, dynamics, nonlinear, universality

1. Introduction

The study of the nonlinear dynamics of structures represents a powerful tool for damage detection. Vibration-based inspection of structural behavior offers an effective tool of nondestructive monitoring. The analysis of the dynamic response of a structure to excitation forces and the monitoring of alterations, which may occur during its lifetime, can be employed as a global integrity-assessment technique to detect, for example, the presence of a crack or play in joints. The damage assessment problem in cracked structures has been extensively studied in the last decade [1–5], highlighting that the vibration-based inspection is a valid method to detect, localize and quantify cracks especially in one-dimensional structures. Dealing with the presence of a crack in a beam, previous studies have demonstrated that a transverse crack can change its state (from open to closed and vice versa) when the structure, subjected to an external load, vibrates [3,4]. As a consequence, a nonlinear dynamic behavior is introduced.

The aim of this chapter is the investigation of the universal behavior in the complex oscillatory behavior of damaged nonlinear structures. In particular, we have focused our attention on a cantilever beam with several breathing transverse cracks and subjected to harmonic excitation perpendicularly to its axis. The method, that is an extension of the high-harmonic analysis presented in [4] to subharmonic and zero-frequency

components, allows us to capture the complex behavior of the nonlinear structure, for example, the occurrence of period doubling, as experimentally observed [5]. The approach is written in terms of the black box interaction formalism, recently developed for high-harmonics (see [6] and Chapter 1), but including subharmonics and multidegrees of freedom, as described in References [7,8].

2. Nonlinear Dynamics of Structures

2.1 The Interaction Box Formalism

Consider a nonlinear structure, having several degrees of freedom and subjected to the multicomponent cause $\{C\}$ of the vibration, that is, a set of harmonic forces/couples with multiple angular frequencies rather than a fundamental angular frequency ω. The effect $\{E\}$, dual to the cause $\{C\}$, that is, the structural displacements (translations and rotations), must satisfy:

$$[M]\{\ddot{E}\} + [D]\{\dot{E}\} + [K]\{E\} + \{B(\{E\})\} = \{C\}, \tag{6.1}$$

where $[M]$, $[D]$, and $[K]$ represent the mass, damping, and stiffness matrices respectively, and $\{B\}$ is the nonlinear component of the box (or structure) [6]. See the Appendix. For free vibrations $\{C\} = \{0\}$.

In general the cause $\{C\}$ can be put in the following form,

$$\{C\} = \sum_{j=0}^{N} \left(\{C_S\}_j \sin j\,\omega t + \{C_C\}_j \cos j\,\omega t\right). \tag{6.2}$$

Assuming as the period of the effect a multiple s of the period of the cause (usually $s = 1$), and according to Fourier analysis, we can write:

$$\{E(t)\} = \sum_{j=0}^{N} \left(\{E_S\}_j \sin j\frac{\omega}{s}t + \{E_C\}_j \cos j\frac{\omega}{s}t\right), \tag{6.3}$$

in which an s different from the unity parameter describes subharmonic generation ([7,8]; $s = 1$ in [4] and [6]) and N should be large enough (theoretically infinite) to reach a good approximation. Introducing the time dependence for $\{E(t)\}$ of Eq. (6.3) into the nonlinear box part $\{B(E)\}$ yields:

$$\{B(E(t))\} = \sum_{j=0}^{N} \left(\{B_S\}_j \sin j\frac{\omega}{s}t + \{B_C\}_j \cos j\frac{\omega}{s}t\right), \tag{6.4}$$

where $\{B_{S,C}\}$ are constants related to $\{E_{S,C}\}$.

Introducing Eqs. (6.2–6.4) into Eq. (6.1) and balancing the harmonics with the same angular frequency, would formally solve the problem, correlating cause and effect. A algebraic system of nonlinear equations is derived in the form of

$$\begin{bmatrix} [K] - \dfrac{j^2\omega^2}{s^2}[M] & -\dfrac{j\omega}{s}[D] \\ \dfrac{j\omega}{s}[D] & [K] - \dfrac{j^2\omega^2}{s^2}[M] \end{bmatrix} \begin{Bmatrix} \{E_S\}_j \\ \{E_C\}_j \end{Bmatrix} = \begin{Bmatrix} \{C_S\}_j \\ \{C_C\}_j \end{Bmatrix} - \begin{Bmatrix} \{B_S(\{E_S\},\{E_C\})\}_j \\ \{B_C(\{E_S\},\{E_C\})\}_j \end{Bmatrix},$$

(6.5a)

or, in compact form:

$$[A(j)]\{E(j)\} = \{C(j)\} - \{B(j)\},$$

(6.5b)

where $j = 0, 1, \ldots, N$.

For a monochromatic single cause:

$$C_{ij} = C\delta_{js}\delta_{ip},$$

(6.6)

p being the node position corresponding to the point where the sinusoidal cause of intensity C is applied.

Each of the N systems in Eqs. (6.5) can be easily solved numerically using an iterative procedure, starting assuming $\{B(j)\} = \{0\}$ and then evaluating $\{B(j)\}$ according to the solutions for $\{E(j)\}$ derived at the previous step, until a satisfactory convergence is reached.

2.2 Application to Cracked Structures

To quantify universal behaviors in nonlinear dynamics of structures, we refer to a multicracked beam. The cracks "breathe" during the vibration and thus cause a variation of the stiffness of the structure, that is, a nonlinearity. Mathematical details are reported in [4, 7] and briefly summarized in the appendix. We consider here just two different and simple cases: a weakly or a strongly damaged structure. An extensive parametrical investigation can be found in [8]. Only in the latter case, the so-called period doubling phenomenon, experimentally observed by Brandon and Sudraud [5], clearly appears. The beam considered here is the same as that described in the mentioned experimental analysis. It is 270 mm long and has a transversal rectangular cross-section of base and height, respectively, of 13 and 5 mm. The material is UHMW-ethylene, with a Young's modulus of 8.61×10^8 N/m^2 and a density of 935 kg/m^3. We have assumed a modal damping of 0.002. It is discretized with 20 finite elements. We have found that values of $s = 4$ and $N = 16$ give a good approximation; that is, for larger values of s and N, substantially coincident solutions are obtained. The first natural frequency of the undamaged structure is $f_u = 10.6$ Hz. A monochromatic cause, a force at the tip, is considered. The effect is assumed to be the tip displacement.

For each of the two considered structures (Figures 6.1a and 6.2a) the time history of the applied force and of the free-end displacement (Figures 6.1b and 6.2b), are shown as well as the zero-, high-, and subharmonic components of the free-end displacement (Figures 6.1c and 6.2c). Obviously, in a hypothetical linear (i.e., here undamaged) structure, the response is linear by definition with only one harmonic component at the same frequency of the monochromatic excitation (Case 0).

Case 1. In the weakly nonlinear structure of Figure 6.1a, the response converges and it appears only weakly nonlinear, as depicted in Figure 6.1b. The harmonic components in the structural response are the zero–one (presence of a negative offset in
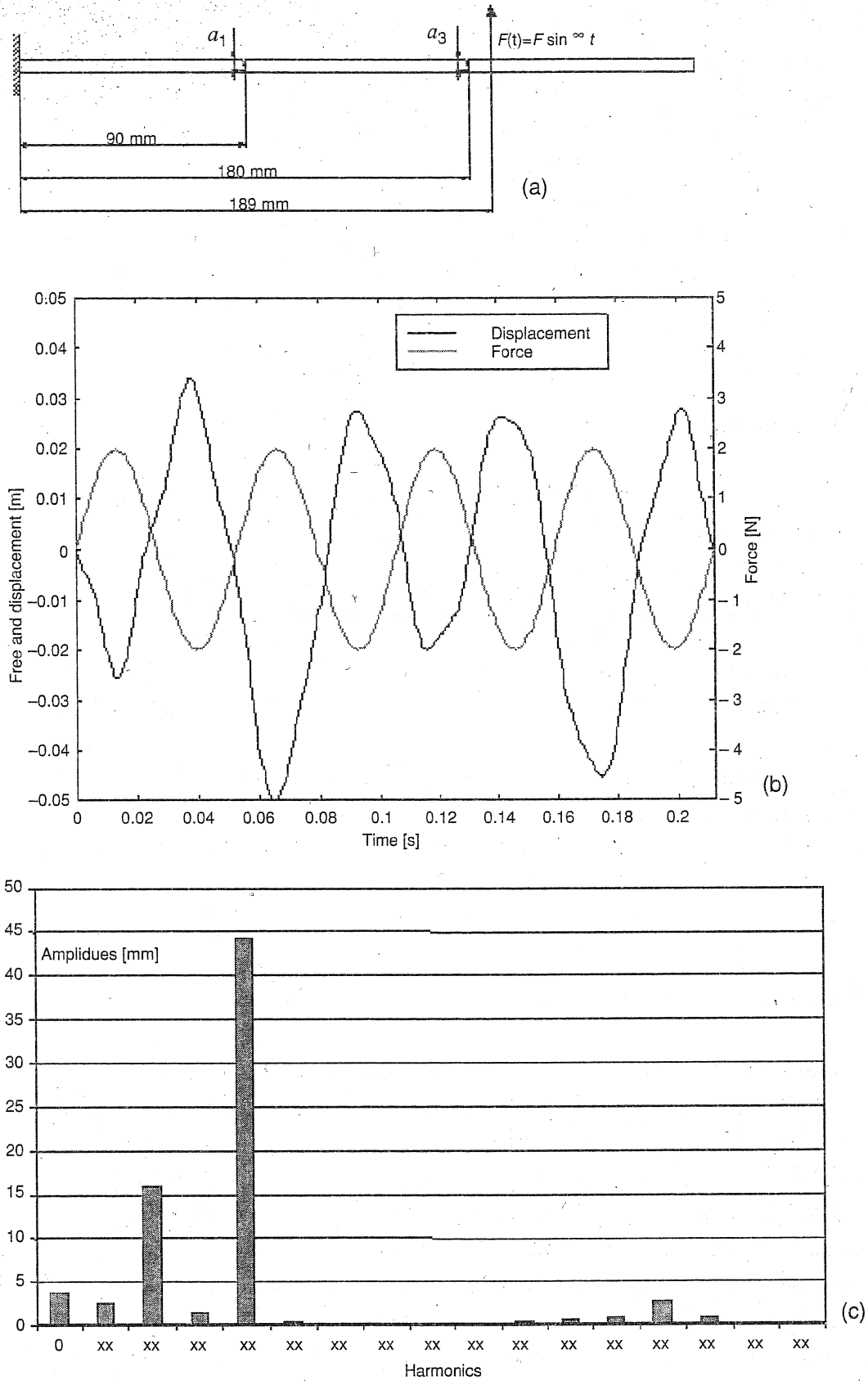
Fig. 6.2. (a) Structure II – Damaged structure and characteristics of the excitation ($a_1 = 4.25\,\mathrm{mm}$, $a_2 = 4.25\,\mathrm{mm}$, $C = F = 2N$, $f = \omega/2\pi = 18.9\,\mathrm{Hz}$); (b) time history of the free end displacement and of the applied force; (c) zero- (offset), sub-, and super-harmonic components for the free end displacement (i.e., $\sqrt{E^2_{S\,20\,j} + E^2_{C\,20\,j}}$ for $j = 0, 1, \dots, 16$).

where E is the Young modulus of the material constituting the finite element; $I = bh^3/12$ is the moment of inertia of its cross-section, having base b and height h; and M and P are the bending moment and shear load acting at the ends of the finite element of length l. The additional energy due to the crack is:

$$W^{(1)} = b \int_0^a \left[\left(K_I^2(x) + K_{II}^2(x) \right) / E' + (1+v) K_{III}^2(x) / E \right] dx, \qquad (A2)$$

where $E' = E$ for plane stress, $E' = E/(1+v)$ for plane strain, and v is the Poisson ratio. $K_{I,II,III}$ are the stress intensity factors for opening, sliding, and tearing-type cracks, of depth a, respectively. Taking into account only bending (i.e., the predominant load):

$$W^{(1)} = b \int_0^a \left[(K_{IM}(x) + K_{IP}(x))^2 + K_{IIP}^2(x) \right] / E' dx, \qquad (A3)$$

with:

$$K_{IM} = \left(6M / bh^2 \right) \sqrt{\pi a} F_I(w)$$

$$K_{IP} = \left(3Pl / bh^2 \right) \sqrt{\pi a} F_I(w) \qquad (A4)$$

$$K_{IIP} = \left(P / bh \right) \sqrt{\pi a} F_{II}(w),$$

where $w = a/h$ and:

$$F_I(w) = \sqrt{2/(\pi w)} \tan(\pi w/2)(0.923 + 0.199(1 - \sin(\pi w/2)^4)) / \cos(\pi w/2)$$
$$F_{II}(w) = (3w - 2w^2)(1.122 - 0.561w + 0.085w^2 + 0.18w^3)/\sqrt{1-w}. \quad (A5)$$

The term $c_{ik}^{(0)}$ of the flexibility matrix $[C_e^{(0)}]$ for an element without a crack can be written as

$$c_{ik}^{(0)} = \frac{\partial^2 W^{(0)}}{\partial P_i \partial P_k} \quad i,k = 1,2 \quad P_1 = P, \quad P_2 = M. \qquad (A6)$$

The term $c_{ik}^{(1)}$ of the additional flexibility matrix $[C_e^{(1)}]$ due to the crack can be obtained as

$$c_{ik}^{(1)} = \frac{\partial^2 W^{(1)}}{\partial P_i \partial P_k} \quad i,k = 1,2 \quad P_1 = P, \quad P_2 = M. \qquad (A7)$$

The term c_{ik} of the total flexibility matrix $[C_e]$ for the damaged element is:

$$c_{ik} = c_{ik}^{(0)} + c_{ik}^{(1)}, \qquad (A8)$$

From the equilibrium condition

$$\left(P_i \ M_i \ P_{i+1} \ M_{i+1} \right)^T = [T] \left(P_{i+1} \ M_{i+1} \right)^T, \qquad (A9)$$

where

$$[T] = \begin{bmatrix} -1 & -l & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}^T .$$ (A10)

Applying the theorem of Enrico Betti (1823–1892), the stiffness matrix of the undamaged element can be written as

$$[K_e] = [T] \left[C_e^{(0)} \right]^{-1} [T]^T ,$$ (A11)

or

$$[K_e] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} ,$$ (A12)

and the stiffness matrix of the cracked element may be derived as

$$[K_{de}] = [T] [C_e]^{-1} [T]^T .$$ (A13)

In order to evaluate the dynamic response of the cracked beam when acted upon by an applied force, it is supposed that the crack does not affect the mass matrix. Therefore, for a single element, the mass matrix can be formulated directly:

$$[M_e] = [M_{de}] = \frac{ml}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} ,$$ (A14)

where m is the mass for unity length of the beam.

Assuming that the damping matrix $[D]$ is not affected by the crack, it can be calculated through the inversion of the modeshape matrix $[\phi]$ relative to the undamaged structure:

$$[D] = \left([\phi]^T \right)^{-1} [d] [\phi]^{-1} ,$$ (A15)

where $[d]$ is the following matrix,

$$[d] = 2 \begin{bmatrix} \zeta_1 \omega_1 M_1 & 0 & 0 & \ldots & 0 \\ 0 & \zeta_2 \omega_2 M_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots \ldots & \ldots & \\ \ldots & \ldots & \ldots \ldots & \ldots & \\ 0 & \ldots & \ldots & 0 & \zeta_n \omega_n M_n \end{bmatrix} ,$$ (A16)

in which ζ_i is the modal damping ratio, ω_i is the ith natural frequency and M_i is the ith modal mass relative to the undamaged beam.

Accordingly, the mass, damping, and stiffness matrices of the structure are deduced by expansion and summation of the element matrices.

Regarding the nonlinearity imposed by the presence of the cracks:

$$\{B(E)\} = \sum_m \left[\Delta K^{(m)} \right] f^{(m)} (\{E\}) \{E\},$$ (A17)

where $[K] + \sum_m [\Delta K^{(m)}]$ is the stiffness matrix of the undamaged beam and $[\Delta K^{(m)}]$ is half of the variation in stiffness introduced when the mth crack is fully open (see [7]). According to this notation, $f^{(m)}(\{E\})$ ranges between -1 and $+1$ and models the transition between the conditions of mth crack fully open and fully closed, depending on the curvature of the corresponding cracked element [4]. Considering the function $f^{(m)}(\{E\})$ as linear versus the curvature of the corresponding cracked element [4, 7] implies

$$f^{(m)}(\{E\}) = \frac{E_{m_k} - E_{m_h}}{\left| E_{m_k} - E_{m_h} \right|_{\max}} = A_m \left(E_{m_k} - E_{m_h} \right), \qquad (A18)$$

where the numerator reports the difference of the rotations for the mth element. Correspondingly:

$$\{B(j)\} = \sum_m \begin{bmatrix} [\Delta K^{(m)}] & [0] \\ [0] & [\Delta K^{(m)}] \end{bmatrix} \left\{ \begin{Bmatrix} H_j^{(m)} \\ K_j^{(m)} \end{Bmatrix} \right\}, \qquad (A19)$$

where

$$H_{ij}^{(m)} = \frac{\Lambda_m}{2} \sum_{j_1, j_2 : j_1 + j_2 = j} \left\{ \left(E_{Sm_k j_1} - E_{Sm_h j_1} \right) E_{Ci j_2} \right.$$
$$+ \left(E_{Cm_k j_1} - E_{Cm_h j_1} \right) E_{Si j_2} \right\} + \frac{\Lambda_m}{2} \sum_{j_1, j_2 : j_1 - j_2 = \pm j}$$
$$\pm \left\{ \left(E_{Sm_k j_1} - E_{Sm_h j_1} \right) E_{Ci j_2} - \left(E_{Cm_k j_1} - E_{Cm_h j_1} \right) E_{Si j_2} \right\}, \quad (A20a)$$

$$K_{ij}^{(m)} = \frac{\Lambda_m}{2} \sum_{j_1, j_2 : j_1 + j_2 = j} \left\{ - \left(E_{Sm_k j_1} - E_{Sm_h j_1} \right) E_{Si j_2} \right.$$
$$+ \left(E_{Cm_k j_1} - E_{Cm_h j_1} \right) E_{Ci j_2} \right\} + \frac{\Lambda_m}{2} \sum_{j_1, j_2 : j_1 - j_2 = \pm j}$$
$$\pm \left\{ \left(E_{Sm_k j_1} - E_{Sm_h j_1} \right) E_{Si j_2} - \left(E_{Cm_k j_1} - E_{Cm_h j_1} \right) E_{Ci j_2} \right\}. \quad (A20b)$$

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