Advances in Science and Technology Vol. 58 (2008) pp 54-59
online at http://www.scientific.net

Online available since 2008/Sep/02

Fractals to Model Hierarchical Biomaterials

Alberto Carpinteri[1,a], Pietro Cornetti[1,b], Nicola Pugno[1,c], Alberto Sapora[1,d]

[1] Politecnico di Torino, Department of Structural Engineering and Geotechnics,
Corso Duca degli Abruzzi 24, 10129 Torino, Italy.

[a]alberto.carpinteri@polito.it, [b]pietro.cornetti@polito.it, [c]nicola.pugno@polito.it,
[d]alberto.sapora@polito.it

Keywords: Biomaterials; hierarchical structures; strength; fractals.

Abstract. Many biological materials exhibit a hierarchical structure over more than one length scale. Understanding how hierarchy affects their mechanical properties emerges as a primary concern, since it can guide the synthesis of new materials to be tailored for specific applications. In this paper the strength and stiffness of hierarchical materials are investigated by means of a fractal approach. A new model is proposed, based both on geometric and material considerations and involving simple recursive formulas.

1. Introduction

Many biological materials are structured in a hierarchical way over more than one length scale [1]. Multiple examples can be given such as bones [2], teeth [3] and shells [4] and: bones, for instance, have seven levels of hierarchy, while sea shells present two or three orders of lamellar structures. These materials, at the most elementary level of structural hierarchy, are composed by hard and strong mineral structures embedded in a soft and tough matrix. In bone the mineral platelets are ~3nm thick, in shells their thickness is of ~300nm, while in tooth crystals are ~15-20nm thick, with a very high slenderness. Understanding how these structures are related to their mechanical properties emerges, hence, as a primary concern, since it may provide guidance on the development of novel materials with unique properties [5].

Hierarchical structures of biomaterials have been recognized to exhibit self-similarity and to be fractal-like [6]. Size effects on apparent mechanical properties due to the fractal nature of material microstructure have been extensively studied [7,8] (see also [9]). If infinite levels of hierarchy are considered, new universal properties (i.e. scale-invariant quantities) having non-conventional or anomalous physical dimensions must be defined [8]. On the other hand, if the hierarchical character is exhibited only over a finite range of scales [10], as for biological materials, new physical considerations can be drawn.

In the present study, the strength and stiffness of hierarchical biomaterials are investigated by means of a fractal approach. The rules of mixture which let estimate the nominal strength of hierarchical materials are firstly presented (Section 2). By exploiting the self-similarity character of the structure, these rules can be condensed in a unique and synthetic manner by means of a fractal approach [11] (Section 3). Finally, a new fractal model based on a multiplicative process [12], which takes into account both geometry and material features, is proposed (Section 4).

2. Strength of hierarchical materials

Let us consider a tensile test on N-hierarchical fibre-reinforced bar. Its cross-section is composed by hard inclusions embedded in a soft matrix and it is represented in Fig. 1. The nominal stress σ_0 could be evaluated by means of a recursive scheme of rules of mixture as [11]:

$$\begin{aligned}
&\sigma_0 = v_1\sigma_1 + (1-v_1)\sigma_{m,1}\\
&\sigma_1 = v_2\sigma_2 + (1-v_2)\sigma_{m,2}\\
&.....\\
&\sigma_i = v_{i+1}\sigma_{i+1} + (1-v_{i+1})\sigma_{m,i+1}\\
&.....\\
&\sigma_{N-1} = v_N\sigma_N + (1-v_N)\sigma_{m,N}
\end{aligned}, \qquad (1)$$

where σ_i and $\sigma_{m,i}$ denote the stresses in the hard and soft phases, respectively, and v_i is the volumetric fraction of inclusions at the i-level (the nanostructure of bones, for instance, shows a mineral to matrix volume ratio in the order of 1 to 2.). The values of the material properties at each level hence depend on those of the preceding levels.
In the case of round (or square)-shaped inclusions at each level, v_i could be expressed in the following form:

$$v_i = n_{h_i}\left(\frac{R_i}{R_{i-1}}\right)^2, \qquad (2)$$

being n_{h_i} the number of inclusions and $R_i \sim \sqrt{A_i}$ their mean average radius. Henceforth, the area of the cross-section of the bar A_0 and its characteristic dimension R_0 will be denoted simply by A and R, respectively.

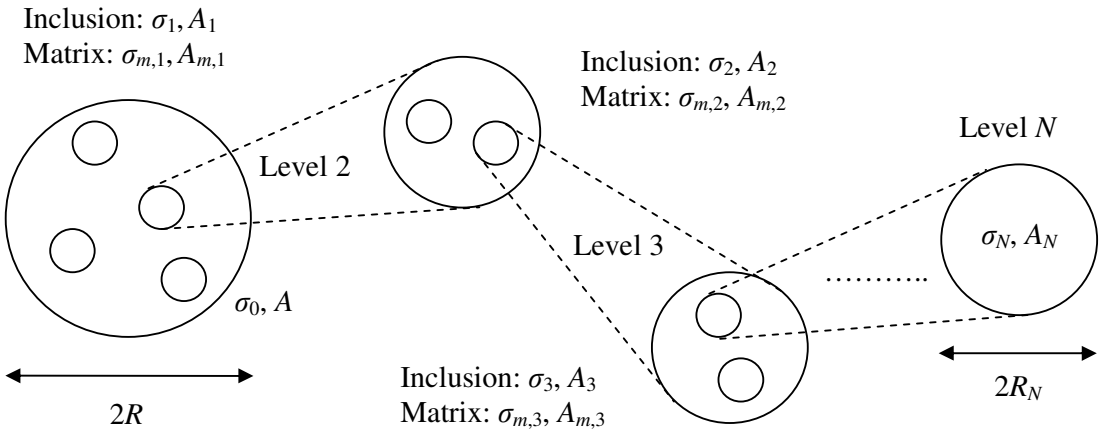

Figure 1. The cross-section of a hierarchical bar.

Note that the rules of mixture (Eq. 1) involve, at each step, only the hard phase (i.e., the inclusions, Fig. 1). Although experimental data [13] show that the properties of the matrix may vary at each level, it is often assumed, for the sake of simplicity, that they do not change, i.e., $\sigma_{m,i} = \sigma_m$. In such a case Eq. 1 can be rewritten via a bottom-up approach as:

$$\sigma_0 = \left(\prod_{i=1}^{N} v_i\right)\sigma_N + \left(1 - \prod_{i=1}^{N} v_i\right)\sigma_m = v_r\sigma_N + (1-v_r)\sigma_m, \qquad (3)$$

where v_r is the total volumetric fraction of inclusions.

3. Fractal model

Natural optimization suggests self-similar structures [14], for which $R_i/R_{i-1} = 1/\delta$ and $n_{h_i} = n_h$; thus, $v_i = v$ and $v_r = v^N$. Thanks to self-similarity, further considerations based on fractal geometry naturally rise up. Particularly, if the hierarchical levels were infinite ($N \to \infty$), the domain of inclusions would result into a fractal set of dimension D:

$$D = \frac{2\ln n_h}{\ln n_h - \ln v} = \frac{\ln n_h}{\ln \delta}, \tag{4}$$

where $0 < D < 2$. See, for instance, the Sierpinski carpet displayed in Fig. 2.
Since a finite range of scales is taken into account, the total volumetric content of inclusions v_r can be modelled as [11]:

$$v_r = v^N = \left(\frac{R_N}{R}\right)^{2-D}. \tag{5}$$

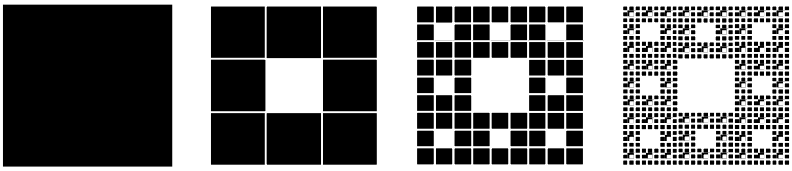

Figure 2. The Sierpinski carpet (D=1.89) at different levels of observation; it corresponds to the cross section of a deterministic hierarchical bar in which the white zones represent the soft matrix, and the black ones are the hard inclusions.

Thus, by substituting Eq. 5 into Eq. 3:

$$\sigma_0 = \left(\frac{R_N}{R}\right)^{2-D}\sigma_N + \left(1 - \left(\frac{R_N}{R}\right)^{2-D}\right)\sigma_m \approx \left(\frac{R_N}{R}\right)^{2-D}\sigma_N, \tag{6}$$

which predicts that the nominal strength σ_0 decreases as the size increases ($R \to \infty$) i.e. “smaller is stronger”. The approximation in Eq. 6 keeps true as long as $\sigma_N >> \sigma_m$, a condition which is usually satisfied [1].

Note that, according to Eq. 6, the stress levels always remain two, whatever is the hierarchical level of the material. The same scaling law is valid for stiffness [11].

4. Multiplicative process model

In the previous section a scaling law for strength in hierarchical materials has been derived, involving the fractal dimension related to the distribution of inclusions (Eq. 4). This model is based on the simplifying hypothesis of identical matrix properties at each scale level (Eq. 3). On the other hand, as pointed out in Section 1, different properties of the matrix are often noticed at different levels. In order to consider also this feature, a new model is proposed, based on a multiplicative process [13], which will be shown to provide a simple and natural extension of Eq. 6.

Let us suppose that the ligament, which is apparently homogeneous at the macro-scale, is divided into n equal parts: the stiffness is allocated such that n_h cells are enriched by a factor Φ, and

n_m cells are depleted by a factor φ. Naturally, $n_h+n_m = n = R^2/R_1{}^2$. If the condition of perfect bonding between inclusions and matrix is satisfied, the same repartition occurs for stresses:

$$A\sigma_0 = A_1\sigma_1 + A_{m,1}\sigma_{m,1} = \left(\frac{n_h}{n}\Phi + \frac{n_m}{n}\varphi\right)A\sigma_0, \tag{7}$$

where A_i now denotes the total area of inclusions at the level i.
The relationship between Φ and φ can be obtained by imposing the condition that the critical applied force must be the same; from Eq. 7, in formulae:

$$\frac{n_h}{n}\Phi + \frac{n_m}{n}\varphi = 1, \tag{8a}$$

and thus

$$\varphi = \frac{n - n_h\Phi}{n_m}. \tag{8b}$$

Since negative values of parameters Φ and φ lack a physical meaning, it can be deduced from Eq. 8b that $\Phi \in (1, n/n_h)$ and $\varphi \in (0,1)$.
At the following level, the problem is renormalized so that the "enriched" cells are structured exactly in the same way:

$$A_1\sigma_1 = A_2\sigma_2 + A_2\sigma_{m,2} = \left(\left(\frac{n_h}{n}\right)^2\Phi^2 + \frac{n_h n_m}{n^2}\Phi\varphi\right)A\sigma_0. \tag{9}$$

Note that this approach is consistent with Eq. 1. At the generic level N, iterating such a procedure yields:

$$F = A\sigma_0 = A_N\sigma_N + \sum_{j=1}^{N} A_{m,j}\sigma_{m,j}, \tag{10}$$

where

$$A_N = \left(\frac{n_h}{n}\right)^N A, \tag{11a}$$

$$\sigma_N = \Phi^N\sigma_0, \tag{11b}$$

and

$$A_{m,j} = \left(\frac{n_h}{n}\right)^j \frac{n_m}{n_h} A, \tag{11c}$$

$$\sigma_{m,j} = \varphi\Phi^{j-1}\sigma_0. \tag{11d}$$

The material properties are hence distributed in a non-homogeneous way to form a hierarchical structure with different characteristics at different length scales. The distribution process here proposed (Eq. 10) is equivalent to assuming that the ratio between the Young's moduli of matrix and inclusions remains constant at each level.
By expressing Eqs. 10a,b as a function of N and equalling the two expressions, the relationship between the nominal strength σ_0 and the strength of inclusions at level N can be obtained:

$$\sigma_0 = \left(\frac{A_N}{A}\right)^{\frac{\log\Phi}{\log(n/n_h)}} \sigma_N \,. \tag{12}$$

Finally, since:

$$\frac{A_n}{A} = \left(\frac{R_N}{R}\right)^{2\frac{\log(n/n_h)}{\log n}}, \tag{13}$$

substituting Eq. 13 into Eq. 12, yields:

$$\sigma_0 = \left(\frac{R_N}{R}\right)^{D^*} \sigma_N \,, \tag{14}$$

where

$$D^* = \frac{2\log\Phi}{\log n} = D\frac{\log\Phi}{\log n_h}, \qquad \Phi \in \left(1, n/n_h\right). \tag{15}$$

Eqs. 14-15 describe the nominal strength dependency not only on the volume fraction of inclusions (geometric effect, by D), but also on the nature of such inclusions (material effect, by Φ). Note that, in the homogeneous case ($\Phi = 1$), Eq. 15 provides $D^* = 0$ and consequently $\sigma_0 = \sigma_N$ (Fig. 3). On the other hand, if the specimen is made only of inclusions ($\Phi = n/n_h$), it is not difficult to demonstrate that $D^* = 2–D$ and the same scaling predicted by Eq. 6 is recovered (Fig. 3).

The present model, by modelling different matrix properties at each level, can hence be considered as an extension of the model presented in Section 3 [11], which is based on purely geometrical considerations.
Under the hypothesis mentioned above, the same scaling occurs clearly for stiffness:

$$E_0 = \left(\frac{R_N}{R}\right)^{D^*} E_N \,, \tag{16}$$

where E_0 and E_N are the Young's moduli of the bar and of the inclusions at level N, respectively, and D^* is provided by Eq. 15.

4. Conclusions

In this work, the strength and hardness of hierarchical materials are investigated. The study has focused on the behaviour of biomaterials subjected to uniaxial loading. Despite these hierarchical structures show a self-similarity character only over a finite range of scales, it is possible to model their scaling properties at each level by means of a fractal approach. Recursive relationships are presented, based on a renormalization group transformation.

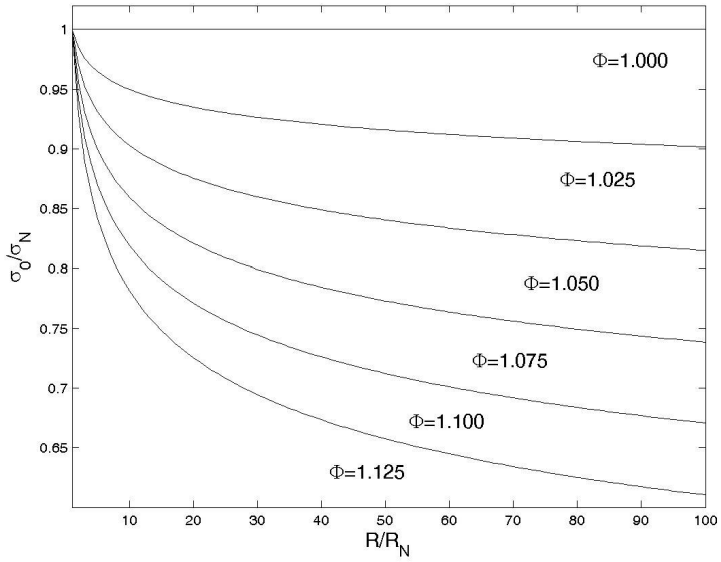

Figure 3. Scaling law of the nominal strength σ_0 for different values of Φ (n = 9, n_h = 8). If Φ = 9/8 = 1.125 the classical Sierpinski carpet is recovered and D^* = 2–(log8/log3) (Eq. 15).

References

[1] H. Gao: International Journal of Fracture 138 (2006), 101.
[2] J.D. Currey: *The Mechanical Adaptations of Bones* (Princeton University Press, NJ 1984).
[3] H. Warshawsky: Anatomical Record 224 (1989), 242.
[4] J.D. Currey: Proceedings of the Royal Society B 196 (1977), 443.
[5] P. Fratzl, R. Wienkamer: Progress in Materials Science 52 (2007), 1263.
[6] R. Lakes: Nature 361 (1993), 511.
[7] A. Carpinteri: Mechanics of Materials, 18 (1994), 89.
[8] A. Carpinteri: International Journal of Solids and Structures 31 (1994), 291.
[9] A. Carpinteri, N. Pugno: Nature Materials 4 (2005), 421.
[10] N. Pugno: Nanotechnology 17 (2006), 5480.
[11] A. Carpinteri, N. Pugno: Philosophical Magazine Letters, in press.
[12] C. Halsey, M. Jensen, L. Kadanoff, I. Procaccia, B. Shraiman: Phys. Review A 33 (1986), 2.
[13] J.H. Brown, G.B. West: *Scaling in biology* (Oxford University Press, Oxford 1999).
[14] H.C. Lee, J. Gurland: Materials science and engineering 33 (1978), 125.