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Fractal fragmentation theory for shape effects of quasi-brittle materials in compression

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The influence of the slenderness (shape effects) of a specimen in compression and of the friction between it and the loading platens on the dissipated energy density and on the compressive strength (strain-softening response) is theoretically and experimentally analysed. The energy dissipated during the process is assumed to be proportional to the area of the free surface of the fragments created under compression. A very general law, describing the energy dissipation in natural as well as in man-made fragmentation phenomena, is herein presented, obtaining, as particular cases, the classical comminution laws (Surface, Volume and Third Comminution theories). As a consequence, the dissipated energy density and the strength for a structural element under compression are obtained, by varying its slenderness under different boundary friction conditions. Finally, a comparison between experimental data and theoretical predictions on shape effects is presented. The influences of the specimen slenderness and friction on dissipated energy density and compressive strength are captured by the proposed model in a satisfactory way.

Notation

A_f	total fracture surface area of fragments
b	specimen basis side
D	fractal exponent
h	specimen height
N_p	total number of fragments
P	cumulative size-distribution function for fragments
p	probability size-distribution function for fragments
r	fragment size
r_{max}	size of the largest fragment
r_{min}	size of the smallest fragment
$V = l^3$	specimen volume under compression
$V_f = l_f^3$	fragmented volume
W	energy dissipated during fragmentation
β	friction-exponent
$\lambda = \frac{h}{b}$	specimen slenderness
σ_C	material strength
$\Psi = \frac{W}{V}$	dissipated energy density under compression

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Introduction

Although the compressive mechanical behaviour of concrete, has been studied by several authors, there is still no complete or systematic treatment, even if many salient aspects have already been emphasised. The most important of these aspects is represented by the phenomenon of strain-softening, that presents different characteristics by varying the test conditions. There are in fact several parameters to be taken into account, and two are the most important: the slenderness of the specimen and the friction between the specimen and the loading platens.

The investigations carried out by Carpinteri *et al.*[1,2] emphasised these aspects both numerically and experimentally. The present investigation deals with this topic from a theoretical point of view based on the fractal fragmentation theory.[3–5]

Several theoretical models have been proposed linking fractals[6,7] to fracture and fragmentation.[8] These models have been recently reviewed by Perfect.[9]

Carpinteri[10] and Carpinteri *et al.*[11,12] used the fractal and multifractal approaches to explain the scaling laws for strength and toughness in the breaking behaviour of disordered materials. Engleman *et al.*[13] applied the maximum entropy method to show that the number-size distribution follows a fractal law for fragments that are not too large. By combining a fractal analysis of brittle fracture with energy balance principles, Nagahama[14] and Yong and Hanson[15] were able to derive a theoretical expression for the fragment size distribution as a function of energy density. Aharony[16] predicted the fragment size distribution from clusters of connected bonds in a cubic lattice using percolation theory. On the other hand, only more recently has the fragmentation theory been applied to the study of compression.[11,17]

The aim of this paper is to evaluate the shape effects on the energy dissipation and strength during the fragmentation of specimens under compression, by varying their slenderness, taking into account friction phenomena. This result is a considerable advancement on the previous knowledge.

Fractal fragmentation theory

After comminution or fragmentation, the cumulative distribution of particles with radius smaller than r is[5,18–20]

$$\mathrm{p}(< r) = 1 - \left(\frac{r_{\min}}{r}\right)^{\mathrm{D}} \tag{1}$$

with D typically comprised between 2 and 3 (e.g. artificially crushed quartz $D = 1{\cdot}89$, disaggregated gneiss $D = 2{\cdot}13$, disaggregated granite $D = 2{\cdot}22$, broken coal $D = 2{\cdot}50$, projectile fragmentation of quartzite $D = 2{\cdot}55$, projectile fragmentation of basalt $D = 2{\cdot}56$, fault gouge $D = 2{\cdot}60$, sandy clays $D = 2{\cdot}61$, terrace sands and gravels $D = 2{\cdot}82$, glacial till $D = 2{\cdot}88$, ash and pumice $D = 3{\cdot}54$).[18]

The related boundary conditions are

$$\mathrm{p}(< r_{\min}) = 0 \tag{2a}$$

$$\mathrm{p}(< r_{\max}) \cong 1 \tag{2b}$$

if $r_{\min} \ll r_{\max}$.

Of course, the complementary cumulative distribution of particles with radius larger than r is

$$\mathrm{p}(> r) = 1 - \mathrm{p}(< r) = \left(\frac{r_{\min}}{r}\right)^{\mathrm{D}} \tag{3}$$

The probability density function $p(r)$ times the interval amplitude dr represents the percentage of particles with radius comprised between r and $r + \mathrm{d}r$. It is provided by derivation of the cumulative distribution function equation (1)

$$\mathrm{p}(r) = \frac{\mathrm{dp}(< r)}{\mathrm{d}r} = \mathrm{D}\frac{r_{\min}^{\mathrm{D}}}{r^{\mathrm{D}+1}} \tag{4}$$

The total fracture surface area is obtained by integration

$$\begin{aligned} A_{\mathrm{f}} &= \int_{r_{\min}}^{r_{\max}} N_{\mathrm{p}}(4\pi r^2)\,\mathrm{p}(r)\mathrm{d}r \\ &= 4\pi N_{\mathrm{p}}\frac{\mathrm{D}}{\mathrm{D}-2}r_{\min}^{\mathrm{D}}\left(\frac{1}{r_{\min}^{\mathrm{D}-2}} - \frac{1}{r_{\max}^{\mathrm{D}-2}}\right) \\ &\cong 4\pi N_{\mathrm{p}}\frac{\mathrm{D}}{\mathrm{D}-2}r_{\min}^{2} \end{aligned} \tag{5}$$

where N_{p} is the total number of particles.

On the other hand, the total volume of the particles is

$$\begin{aligned} V_{\mathrm{f}} &= \int_{r_{\min}}^{r_{\max}} N_{\mathrm{p}}\left(\frac{4}{3}\pi r^3\right)\mathrm{p}(r)\mathrm{d}r \\ &= \frac{4}{3}\pi N_{\mathrm{p}}\frac{\mathrm{D}}{3-\mathrm{D}}r_{\min}^{\mathrm{D}}\left(r_{\max}^{3-\mathrm{D}} - r_{\min}^{3-\mathrm{D}}\right) \\ &\cong \frac{4}{3}\pi N_{\mathrm{p}}\frac{\mathrm{D}}{3-\mathrm{D}}r_{\min}^{\mathrm{D}}r_{\max}^{3-\mathrm{D}} \end{aligned} \tag{6}$$

If we assume a material 'quantum' of size $r_{\min}$ = constant[5,19,21–23] and a hypothesis of self-similarity, that is, $r_{\max} \propto \sqrt[3]{V_{\mathrm{f}}}$,[24] the energy W dissipated to produce the new free surface in the comminution process, which is proportional to the total surface area A_{f},[25,26] can be obtained eliminating the particle number from equations (5) and (6)

$$W \propto A_{\mathrm{f}} \propto V_{\mathrm{f}}^{\mathrm{D}/3} \tag{7}$$

and represents an extension of the Third Comminution theory, where $W \propto V_{\mathrm{f}}^{2{\cdot}5}$.[27] The extreme cases contemplated by equation (7) are represented by $D = 2$, surface theory,[28,29] when the dissipation really occurs on a surface ($W \propto V_{\mathrm{f}}^{2/3}$), and by $D = 3$, volume theory,[29,30] when the dississipation occurs in a volume ($W \propto V_{\mathrm{f}}$). The experimental cases of comminution are usually intermediate, as well as the size distribution for concrete aggregates due to Füller.[31] On the other hand, concrete aggregates frequently are a product of natural fragmentation or artificial comminution. If the material to be fragmented is concrete, we have therefore a double reason to expect a fractal response.

The energy dissipation occurs on a two-dimensional surface, rather than on a morphologically fractal set. On the other hand, the distribution of particle size follows a power-law, the number of infinitesimal particles tending to infinity.

The fundamental assumptions of material 'quantum' and of self-similarity can be derived from the more general hypothesis that the energy dissipation must occur in a fractal domain comprised, in any case, between a surface and a volume.[19–20]

If we assume $D < 2$, from equation (5) we have

$$A_{\mathrm{f}} \cong 4\pi N_{\mathrm{p}} \frac{\mathrm{D}}{2-\mathrm{D}} r_{\min}^{\mathrm{D}} r_{\max}^{2-\mathrm{D}} \quad (8)$$

equation (6) is still valid and then equation (7) becomes

$$W \propto A_{\mathrm{f}} \propto \frac{V_{\mathrm{f}}}{r_{\max}} \quad (9)$$

From equation (9) we obtain $r_{\max} = \text{constant}\cdot\sqrt[3]{V_{\mathrm{f}}}$, if the dissipation is assumed to be proportional to $V_{\mathrm{f}}^{2/3}$ even when $D < 2$. The self-similarity assumption has a

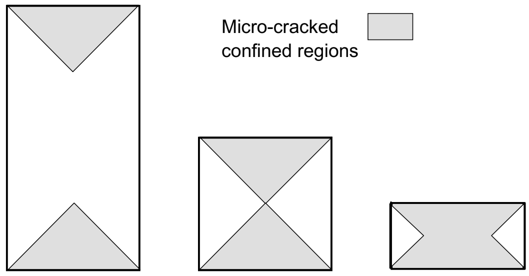

Fig. 1. Micro-cracked confined regions for friction tests

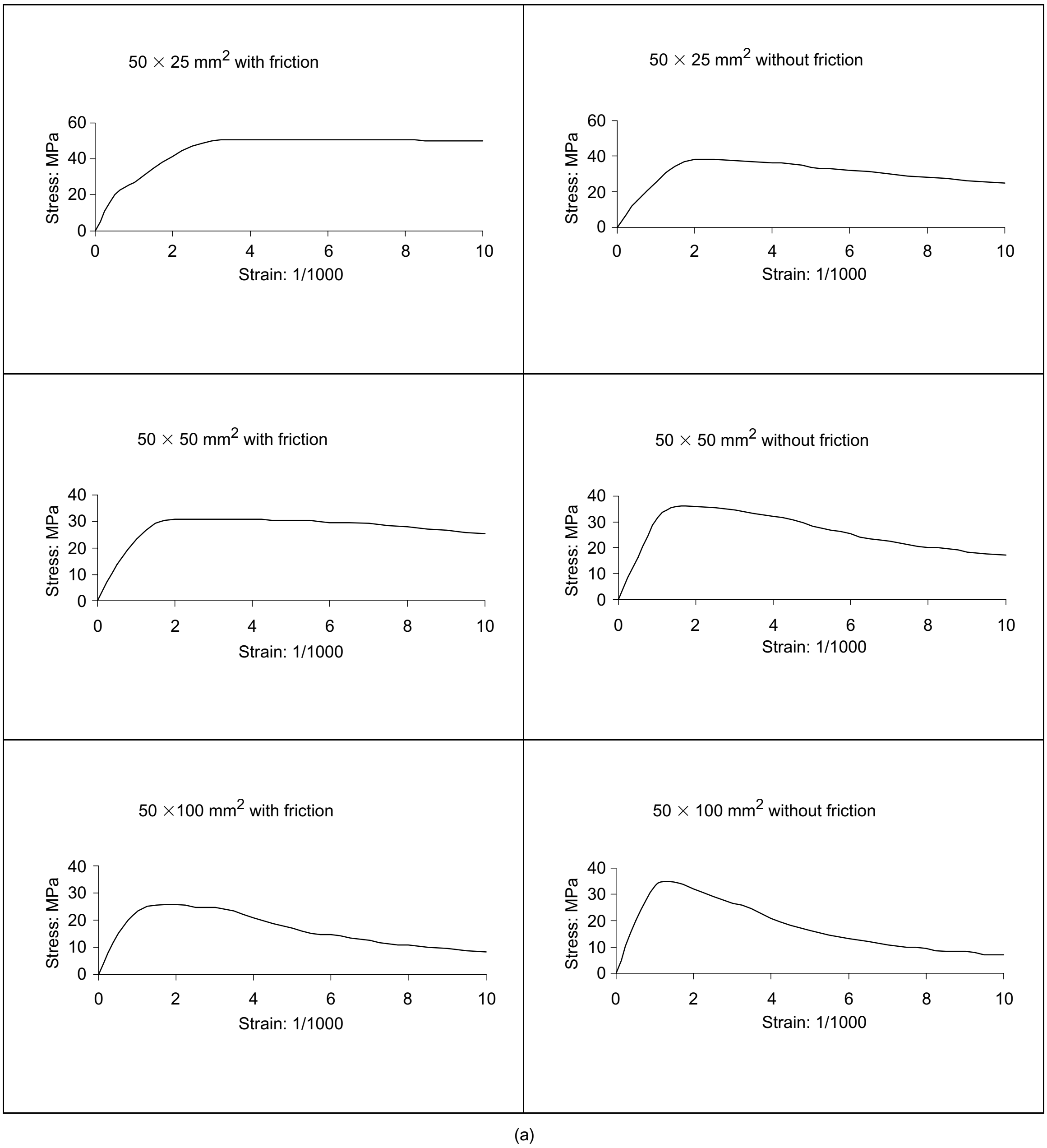

Fig. 2. Stress–strain experimental curves: (a) specimen with side-length of 50 mm; (b) specimen with side-length of 100 mm; (c) specimen with side-length of 150 mm

statistical nature:[24] the larger the fragmented volume, the larger the largest fragment.

If we assume $D > 3$, from equation (6) we have

$$V_f \cong \frac{4}{3}\pi N_p \frac{D}{D-3} r_{min}^3 \quad (10)$$

equation (5) is still valid and then equation (7) becomes

$$W \propto A_f \propto \frac{V_f}{r_{min}} \quad (11)$$

From equation (11) we obtain r_{min} = constant, if the dissipation is assumed to be proportional to V_f even when $D > 3$. The material quantum has been experimentally observed.[19,20,22]

Shape effects

A specimen under compression will be fragmented following the cumulative particle size distribution of equation (1) with D around 2.[17,20]

The fragmented volume $V_f = l_f^3$ and the volume of the specimen under compression $V = l^3$ are not necessarily conincident, so that a power-law relation is assumed[5]

$$l_f \propto l^\beta \quad (12)$$

In the extreme cases, the fragmented volume is independent of the specimen volume so that the exponent β is equal to zero, or they are directly proportional so that β is equal to 1. The exponent β permits to model

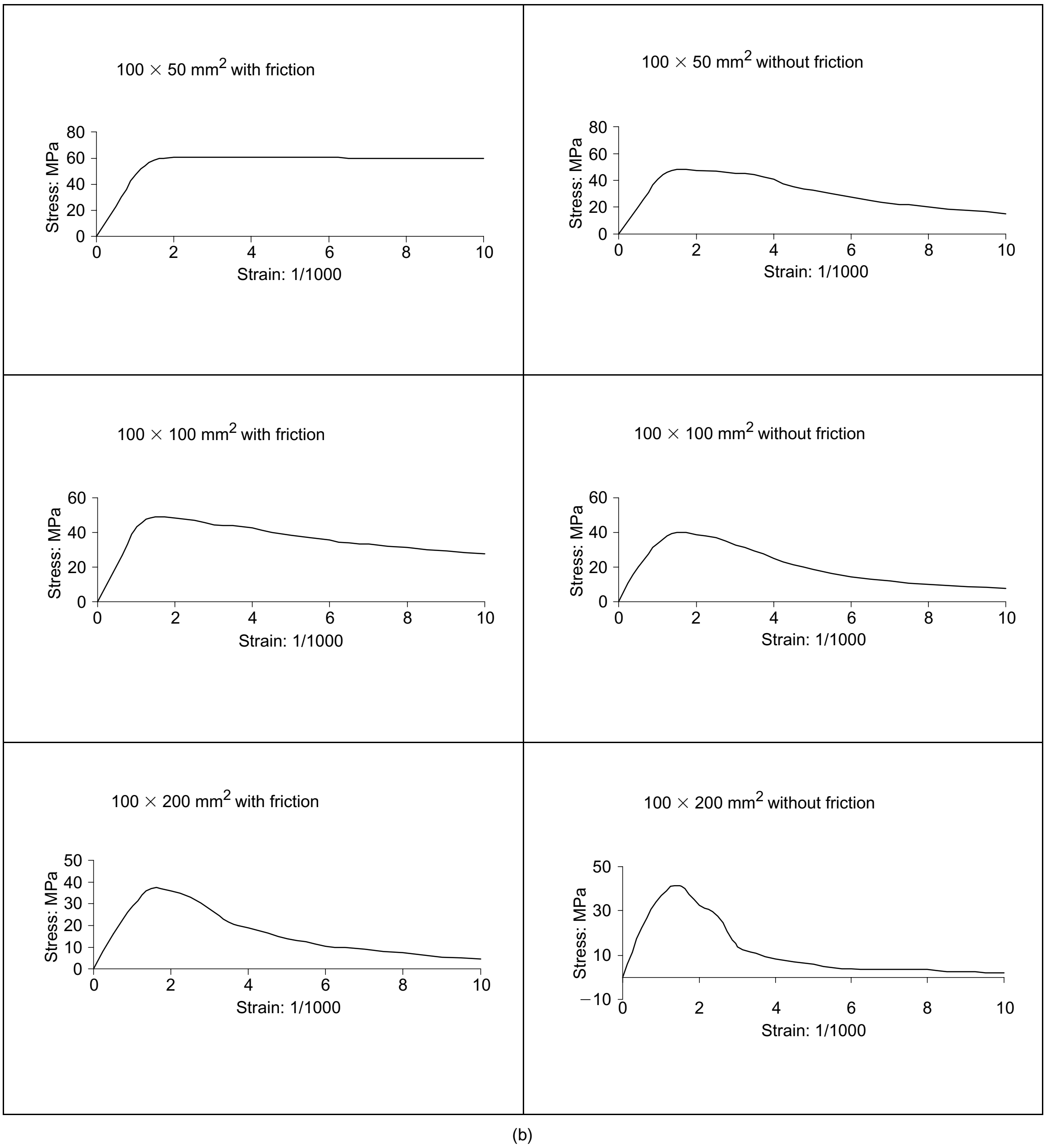

Fig. 2. continued

the friction between loading platens and specimen. Only when there is not any friction, the more intuitive hypothesis of direct proportionality between fragmented volume and specimen volume can be assumed. As a matter of fact, the frictional shearing stresses acting at the interface produce triaxially-confined regions near the bases where a multitude of microcracks propagate.[1,2] In other words, the micro-cracked confined region, that is, the fragmented volume, will be constant varying the slenderness (Fig. 1). For these reasons, we expect larger values of β for frictionless tests than for friction ones.

Considering specimens with constant basis area $A = b^2$, the specimen slenderness λ (height h over basis side b) can be obtained as

$$\lambda = \frac{V}{V_{\lambda=1}} = \frac{b^2 h}{b^3} = \frac{h}{b}$$

and, from equations (7) and (12), we can evaluate the relative dissipated strain energy density $\Psi = W/V$ during the compression of the specimen as a function of its slenderness

$$\frac{\Psi}{\Psi_{\lambda=1}} = \lambda^{\frac{\beta D-3}{3}} \tag{13}$$

Based on equation (13), the shape effects on the

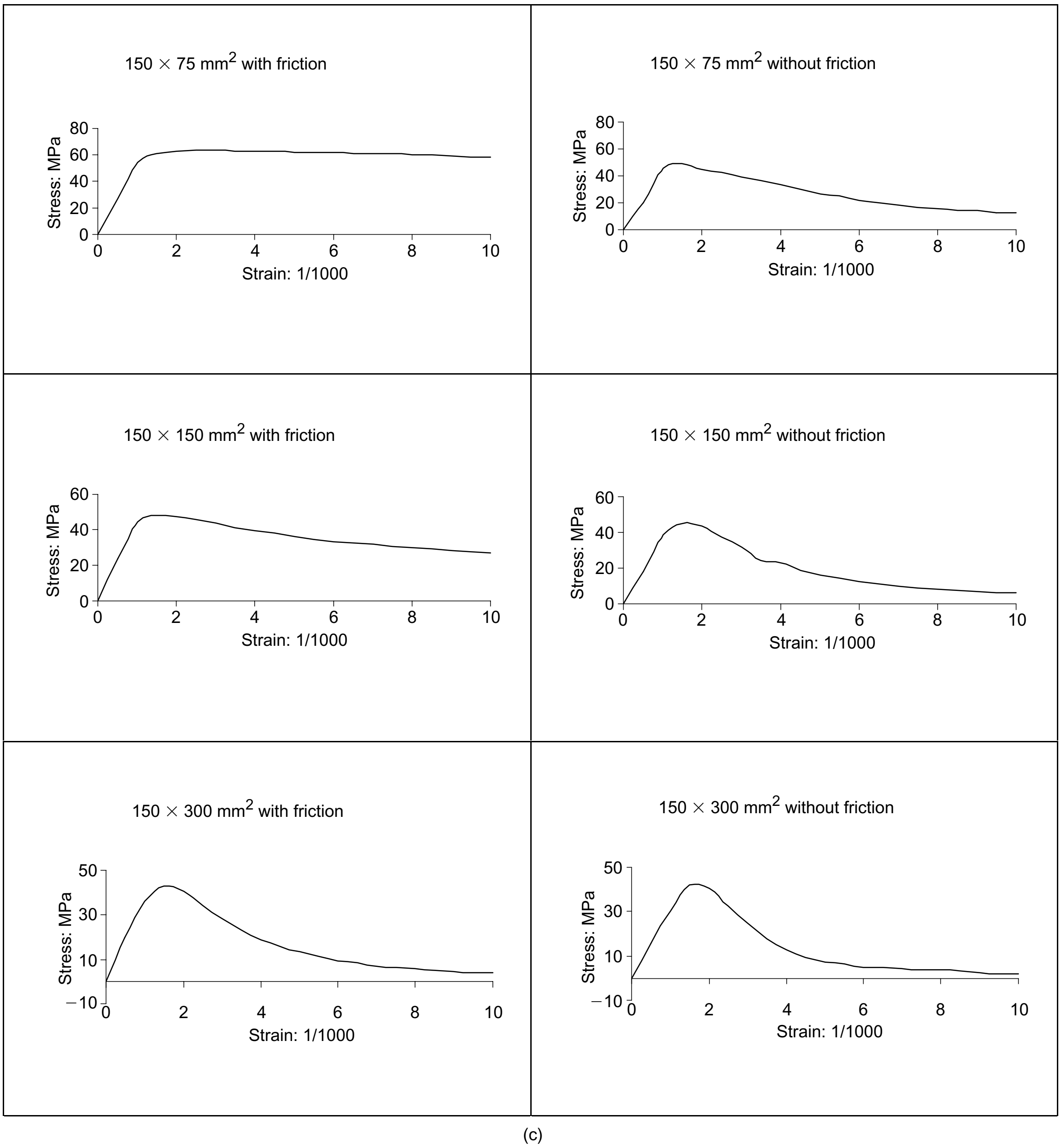

*Fig. 2. (*continued*)*

compressive strength σ_C can be extimated assuming $\Psi \propto \sigma_C^2$. This hypothesis is only a rough assumption to have a first estimation of the shape effect on the compressive strength. It corresponds to assume that the pre-peak area under the stress–strain curve, that is, $1/(2E)\sigma_C^2 \propto \sigma_C^2$, is proportional to the post-peak area. So, that

$$\frac{\sigma_C}{\sigma_C(\lambda = 1)} = \lambda^{\frac{\beta D - 3}{6}} \tag{14}$$

Equations (13) and (14) represent the two fundamental shape effect laws, based on the developed fractal fragmentation theory, where D is close to 2 and β can be considered a best-fit parameter, expected to be larger for frictionless tests than for friction ones.

Experimental assessment

In this section, a comparison between the experimental[1,2] and the theoretical predictions of equation (13) is presented. The comparison regards prismatic concrete specimens (normal strength concrete with σ_C around 40 MPa) with a square basis (50×50, 100×100, 150×150 mm^2), three different slendernesses (0·5, 1·0, 2·0), with or without friction between the specimen itself and the loading platens, for a total of 18 cases (Fig. 2).

The friction condition is represented by the direct contact between specimen and platens, since the shearing stresses at the interface arise in opposition to the lateral Poisson's expansion of the specimen. On the other hand, the introduction of Teflon layers between

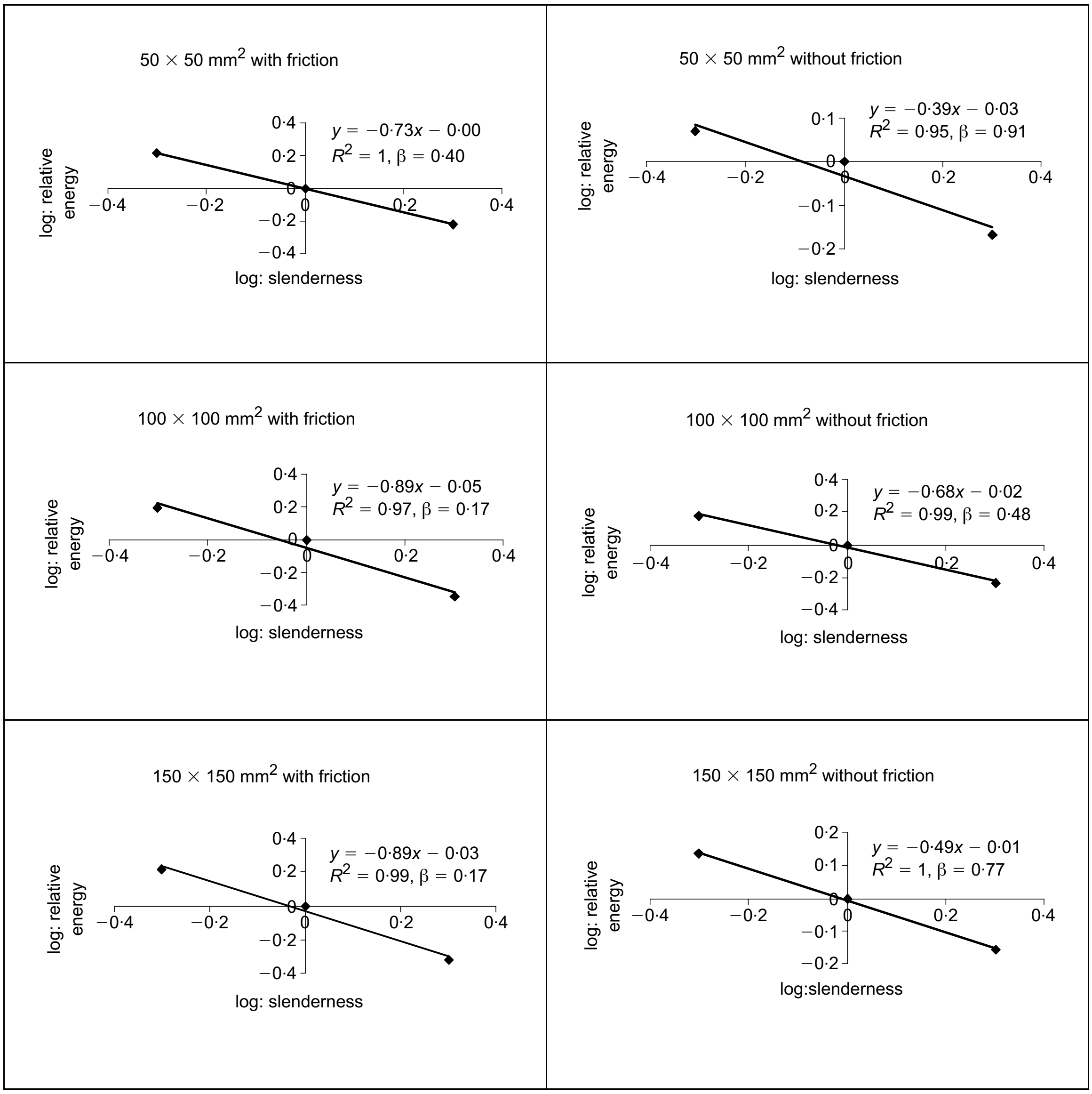

Fig. 3. Dissipated energy density plotted against specimen slenderness. Comparison between theoretical straight line (equation (13) with best-fit parameter β) and experimental points (bi-logarithmic diagrams: $y = \log\Psi/\Psi_{\lambda=1}$ against $x = \log\lambda$)

the specimen and the loading platens allows for the lateral expansion of the material; as a consequence, the shearing stresses at the interface become negligible (the friction coefficient in that case is close to 0·01). For more experimental details see Reference 2.

Although the verification of the theoretical work is based on limited data, equations (13) and (14), described by a straight line in a bi-logarithmic diagram, are experimentally confirmed (Figs 3 and 4). For each curve, the best-fit parameter β has been reported. The roughness of the assumption $\Psi \propto \sigma_C^2$ is the reason for which we derive different values of β for the energy dissipated and the strength.

As expected, the best-fit parameters β are smaller for friction tests than for the corresponding frictionless ones.

Conclusions

A very general law (equation (7)) describing the energy dissipation in natural or man-made fragmentation phenomena has been herein presented. It has been applied to the study of compression. As a consequence, the very simple shape effect laws of equations (13) and (14), based on the developed fractal fragmentation theory, can be used to predict the slenderness and friction influences on the dissipated energy density and compressive strength of quasi-brittle materials under compression.

The analysis of the results presented in the paper shows a satisfactory correspondence between the theoretical predictions and the experimental data. The experimental ductility and compressive strength

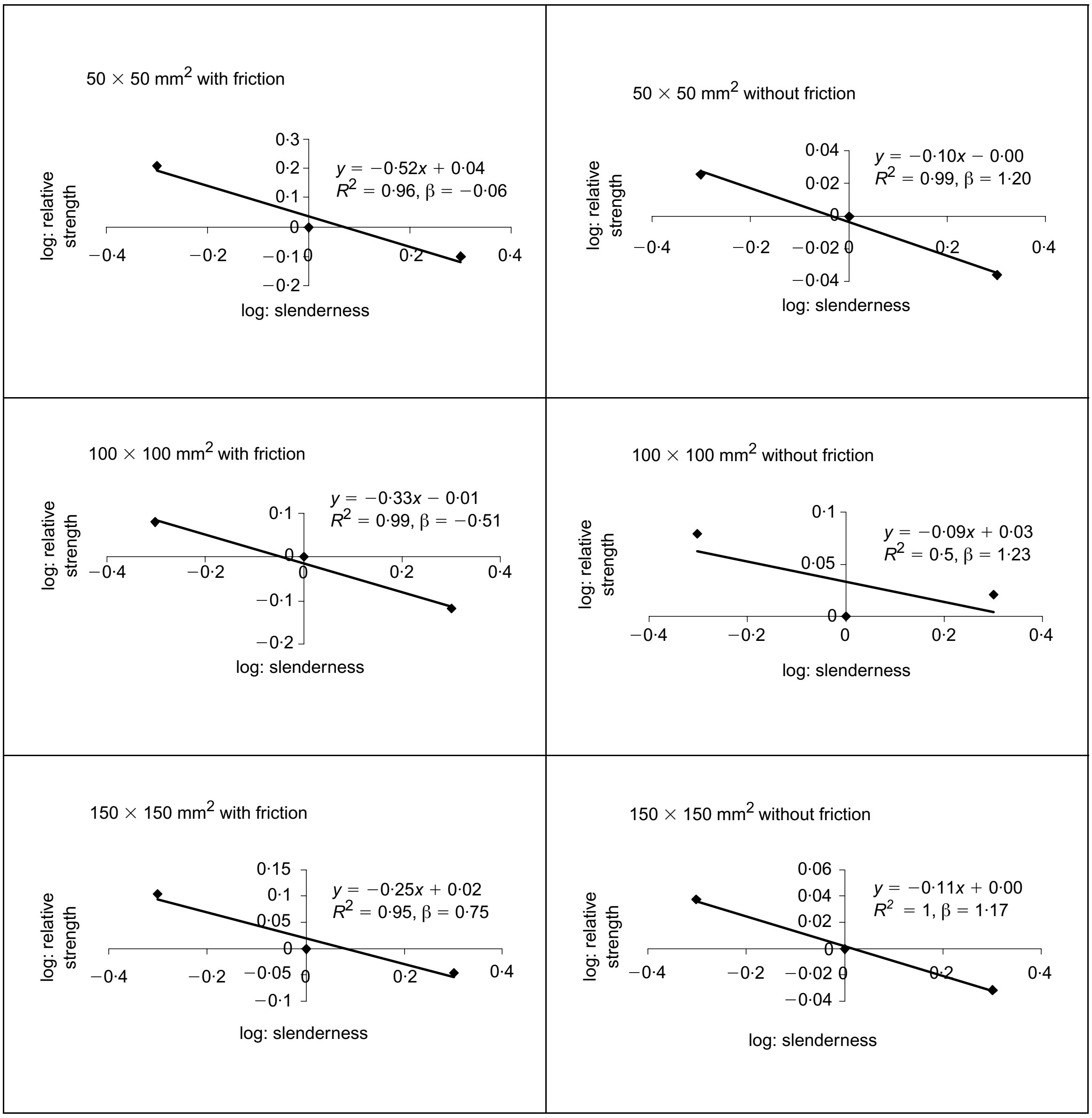

Fig. 4. Compressive strength plotted against specimen slenderness. Comparison between theoretical straight line (equation (14) with best-fit parameter β) and experimental points (bi-logarithmic diagrams: $y = \log\sigma_C/\sigma_C(\lambda = 1)$ against $x = \log\lambda$)

increments with the specimen slenderness decrement, as well as the friction influence, are quantitatively captured by the proposed approach.

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Discussion contributions on this paper should reach the editor by 1 June 2003