

## Velcro® nonlinear mechanics

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In this letter a Velcro® nonlinear mechanics is presented. In particular, a calculation of the “elastic strength” of hooks with friction is derived. The author quantifies, as the intuition and Velcro® material suggest, that hooks (and loops) allow reversible strong attachment, finding elastic plastic or hyperelastic nonlinear behaviors, as a function of the competition between friction and finite displacements. Thus, the author presents here a Velcro® nonlinear mechanics to design and optimize hooked systems. © 2007 American Institute of Physics. [DOI: 10.1063/1.2715478]

Burdock plants [Fig. 1(a)] inspired in 1941 the Swiss engineer de Mestral during a daily walk in the Alps; ten years later Velcro® was patented.<sup>1</sup> Velcro® mechanics has recently been observed in wood<sup>2</sup> and is expected to play a fundamental role in many hooked systems, e.g., in the grip of *evarcha arcuata* spiders<sup>3</sup> [Fig. 1(b)]. The aim of this letter is thus the development of a basic, even if by force nonlinear, Velcro® mechanics to design and optimize hooked systems.

Consider an elastic movable hook with friction, see Fig. 2(a). The hook has a radius of curvature  $R$ , moment of inertia  $I$  and Young’s modulus  $E$ . The hook center is chosen as the origin of the reference system  $x=R \cos \vartheta$ ,  $y=R \sin \vartheta$ , and thus the generic position along the hook is described by the angle  $\vartheta$ . The contact between hook and substrate takes place at a point designed by  $\vartheta_C$ , where a vertical (along  $y$ ) force  $F$  is imposed. Accordingly, the bending moment along the hook is  $M(\vartheta)=-FR(\cos \vartheta-\cos \vartheta_C)$  for  $\vartheta < \vartheta_C$  and  $M(\vartheta)=0$  for  $\vartheta \geq \vartheta_C$ . As a first approximation the elastic transversal displacement  $u$  must satisfy the classical elastic line equation  $d^2u/ds^2 \approx -M/(EI)^4$ , where  $s=R\vartheta$  denotes the curvilinear coordinate. For our circular geometry  $d^2u/d\vartheta^2 \approx -M(\vartheta)R^2/(EI)$ . Imposing the boundary conditions  $u(\vartheta=0)=du/d\vartheta(\vartheta=0)=0$  we find the hook elastic rotation and displacement for  $\vartheta < \vartheta_C$  in the following forms:

$$u(\vartheta) \approx \frac{FR^3}{EI} \left( -\cos \vartheta_C \frac{\vartheta^2}{2} - \cos \vartheta + 1 \right), \quad (1a)$$

$$\alpha(\vartheta) = \frac{du(\vartheta)}{Rd\vartheta} \approx \frac{FR^2}{EI} (-\cos \vartheta_C \vartheta + \sin \vartheta). \quad (1b)$$

The hook portion described by  $\vartheta \geq \vartheta_C$  undergoes only a rigid rotation-translation with respect to the movable clamp.

The contact angle  $\vartheta_C$  for a rigid hook can be derived imposing  $\vartheta_C = \pi/2 \mp \varphi$ , where  $\varphi$  is the friction angle between hook and substrate and the different signs correspond to the two different relative moving directions. However, we must impose the nonlinear condition taking into account the hook compliance in order to derive the hook elastic strength. Accordingly

$$\vartheta_C = \pi/2 \mp \varphi + \alpha(\vartheta_C). \quad (2)$$

The hook elastic strength can be estimated imposing the critical condition of detachment  $\vartheta_C = \pi$  in Eq. (2) [Fig. 2(b)],

corresponding to  $\alpha(\vartheta_C = \pi) = \pi/2 + \varphi$ ; thus, from Eq. (1b), we derive the maximum attachment force as

$$F_h = \frac{(\pi/2 + \varphi)EI}{\pi R^2}. \quad (3)$$

If a number of (active) hooks per unit area  $\rho_h = m/(\pi(2R)^2)$  is present, corresponding to an equivalent number  $m$  of hooks per clamp, the corresponding nominal strength will be

$$\sigma_h = \rho_h F_h = \frac{m(\pi/2 + \varphi)EI}{4\pi^2 R^4} = \frac{m(\pi/2 + \varphi)E}{16\pi(R/r)^4}, \quad (4)$$

where  $r$  is the radius of an equivalent circular cross section, defined imposing  $I = \pi r^4/4$ . For example, considering  $m=10$ ,  $\varphi=0$ ,  $E=10$  GPa, and  $R/r=10$  corresponds to  $\sigma_h \approx 0.3$  MPa, incidentally (the main contribute to spider adhesion is due to van der Waals forces) comparable to the adhesive strength of *evarcha arcuata* spiders.<sup>3</sup> To have an idea about such a value, note that we are under an atmospheric pressure of  $\sim 0.1$  MPa, that thus would represent the strength of a (ideal) suction cup.

To attach the hook, a hauling angle  $\vartheta > \pi/2$  is required, since  $\vartheta = \pi/2$  represents a bifurcation point. The maximum force for hooking is derived imposing the critical condition  $\vartheta_C = \pi/2^+$  in Eq. (2) [Fig. 2(c)], corresponding to  $\alpha(\vartheta_C = \pi/2) = -\varphi$ ; thus, from Eq. (1b), we derive

$$f_h = -\frac{\varphi EI}{R^2}. \quad (5)$$

Consequently the ratio,

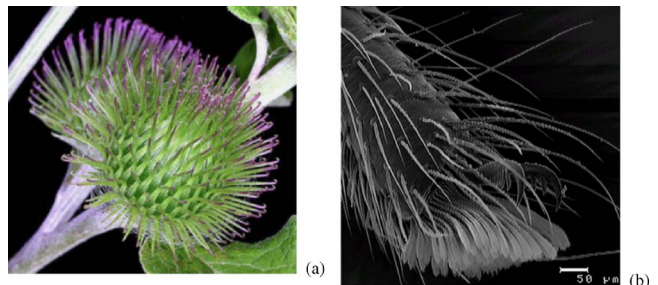


FIG. 1. Hooks of a Burdock plant (a) (adapted from the web) and those observed in the *Evarcha Arcuata* spiders (b) (adapted from Ref. 3).

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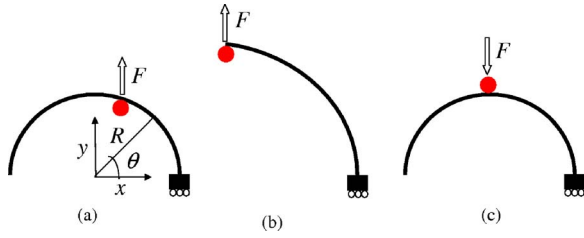


FIG. 2. Elastic hook and loop in contact with friction. Conditions of interlocking (a), ultimate “elastic strength” (b), and hauling (c).

$$\mu = \frac{F_h}{|f_h|} = \frac{1}{\pi} + \frac{1}{2\varphi}, \quad (6)$$

is expected to be very large ( $\mu(\varphi \rightarrow 0) \rightarrow \infty$ ), and thus strong and “reversible” adhesion is expected in hooked materials, as observed in Velcro®.

The force carried by one hook scales as  $F_1 \equiv F_h \propto r^4/R^2$ , thus the bending, tensile, and nominal stresses in the hook must scale as  $\sigma_b \propto r/R$ ,  $\sigma_t \propto (r/R)^2$ , and  $\sigma_h \propto (r/R)$ ,<sup>4</sup> respectively. Accordingly, size effects<sup>5</sup> can be predicted. For example, splitting up the contact into  $n$  subcontacts, i.e.,  $R \rightarrow R/\sqrt{n}$ , would result in a force  $F_n = n^\beta F_1$  with  $\beta=0$  if  $r \propto R$  but  $\beta=2$  if  $r=\text{const}$ . Thus, for  $\beta>0$ , subcontacts are found to be safer, as observed in Nature,<sup>6</sup> even if the hook will be higher stressed and its mechanical strength will impose a lower bound to the radius of the smallest hook.

The nonlinear vertical displacement-force curve for the friction elastic hook can also be computed: the contact angle  $\vartheta_C = \vartheta_C(F)$  can be obtained introducing Eq. (1b) into Eq. (2) and solving with respect to  $\vartheta_C$ ; the displacement  $u = u(F)$  is derived introducing such a relationship into Eq. (1a); thus, the vertical displacement  $\delta(F) = u(F) \sin(\vartheta_C(F))$  can be evaluated. Let us consider the asymptotic solution around the unloaded configuration described, according to Eq. (2), by  $\vartheta_C = \pi/2 - \varphi + \varepsilon(F)$  with  $\varepsilon \rightarrow 0$ . Following the described procedure we find:

$$\delta(F) \approx \frac{FR^3}{EI} [g(\varphi) + h(\varphi)\varepsilon(F)], \quad \varepsilon(F) \approx \frac{FR^2}{EI} f(\varphi), \quad (7a)$$

$$g(\varphi) = \sin \varphi \cos \varphi \left( \frac{1}{2} \left( \frac{\pi}{2} - \varphi \right)^2 - 1 \right) + \cos \varphi, \quad (7b)$$

$$h(\varphi) = \cos^2 \varphi \left( 1 - \frac{1}{2} \left( \frac{\pi}{2} - \varphi \right)^2 \right) + \sin \varphi \cos \varphi \left( \frac{\pi}{2} - \varphi \right), \quad (7c)$$

$$f(\varphi) = \cos \varphi - \sin \varphi \left( \frac{\pi}{2} - \varphi \right). \quad (7d)$$

Since  $f(\varphi) > 0$  ( $f(\varphi) < 0$  for  $0 < \varphi < \pi/2$ ;  $f(\varphi=0)=1$  and  $f(\varphi=\pi/2)=0$ ) the sign of  $h(\varphi)$  denotes elastic plastic ( $h(\varphi) > 0$ ) or hyperelastic ( $h(\varphi) < 0$ ) constitutive laws. Accordingly, for  $\varphi < \varphi_C \approx \pi/39$ , we find a hyperelastic behavior, whereas for  $\varphi > \varphi_C$  elastic plasticity is observed. For example, for  $\varphi=0$  we have hyperelasticity described by  $\delta(F) \approx FR^3/(EI)[1 + (1 - \pi^2/8)FR^2/(EI)]$ . Evidently, hyperelasticity is activated by the nonlinear behavior imposed by Eq. (2) also in the absence of friction (for a linear frictionless hook  $\vartheta_C = \pi/2$ ), whereas friction tends to increase the nonlinearity in the opposite way, making the behavior elastic plastic for  $\varphi > \varphi_C$ ; note that for  $\varphi = \varphi_C$  the nonlinear hook behaves as linear.

Finally, the work of adhesion  $2\gamma_h$  per unit area, an index of the adhesive toughness can be computed according to

$$2\gamma_h = \rho_h \int_0^{F_h} \delta(F) dF = \left( \frac{1}{2} + \kappa \right) \sigma_h \delta(F_h), \quad (8)$$

where  $\kappa=0$  for linear systems. For example, considering  $\kappa=0$ ,  $\sigma_h=0.3$  MPa, and  $\delta(F_h) \approx R=1$   $\mu\text{m}$  yields an estimation of  $\gamma_h \approx 0.3$  N/m.

Summarizing, we have presented a basic (according to the simplest elastic line equation considered), but by force nonlinear, Velcro® mechanics, that could help in the design and optimization of hooked adhesive systems, with respect to their strength, toughness, and constitutive law.

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