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A generalized Paris' law for fatigue crack growth

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Abstract

An extension of the celebrated Paris law for crack propagation is given to take into account some of the deviations from the power-law regime in a simple manner using the Wöhler SN curve of the material, suggesting a more general "unified law". In particular, using recent proposals by the first author, the stress intensity factor $K(a)$ is replaced with a suitable mean over a material/structural parameter length scale Δa, the "fracture quantum". In practice, for a Griffith crack, this is seen to correspond to increasing the effective crack length of Δa, similarly to the Dugdale strip-yield models. However, instead of including explicitly information on cyclic plastic yield, short-crack behavior, crack closure, and all other detailed information needed to eventually explain the SN curve of the material, we include directly the SN curve constants as material property. The idea comes as a natural extension of the recent successful proposals by the first author to the static failure and to the infinite life envelopes. Here, we suggest a dependence of this fracture "quantum" on the applied stress range level such that the correct convergence towards the Wöhler-like regime is obtained. Hence, the final law includes both Wöhler's and Paris' material constants, and can be seen as either a generalized Wöhler's SN curve law in the presence of a crack or a generalized Paris' law for cracks of any size.

Keywords: Fatigue; Crack propagation; Fracture; Paris' law; SN curves

1. Introduction

Fatigue life prediction is still very much an empirical art rather than a science, despite being a relatively old subject having nearly 150 years of history (Wöhler, 1860), described

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in a number of books or review papers (see for example, Schutz, 1996; Cui, 2002; Fleck et al., 1994; Frost et al., 1974; Suresh, 1998, to name a few). In the old days, SN curves were measured and their empirical nature generally maintained even when simplified equations like the Basquin power law (Basquin, 1910) emerged, or when recognizing simpler rules for the "fatigue limit", or various other effects on the basic fatigue SN curve, like the mean stress, roughness, notch, dimension. This was generally driven by the need to have engineering rules still used when designing against fatigue with the "safe-life" approach, i.e., virtually for infinite life.

With the advent of fracture mechanics, a more ambitious task was undertaken, i.e., to predict, or to at least understand, the propagation of cracks. It clearly emerged that the propagation "speed" was far from being constant in time: generally, it was clear that the crack advance was larger for increasing stress amplitudes, but also for larger cracks, until the pioneering work of Paris et al. (1961, 1963), who suggested to use the Irwin stress intensity factor (more precisely, its range ΔK) to characterize the rate of crack advance per cycle, $\mathrm{d}a/\mathrm{d}N$, since many data collapsed in a single power law in the $\mathrm{d}a/\mathrm{d}N$–ΔK diagram. Since then, much work has been done to understand more of Paris' law and its deviations, but we are still far from a complete understanding (see review by Newman, 1998).

In a sense, most of the present design processes still use the empirical approaches of the "pre-Paris" era, particularly for "long-lives" (high cycle fatigue (HCF)), for which a "damage tolerant" approach is very difficult to implement, since cracks grow very slowly and are not detected for large part of their life. It is still difficult today, even with the most advanced models, material data information, and computational capabilities, to derive an entire SN curve from integration of crack propagation laws, particularly because of various regimes of deviations from the standard Paris law regime, in particular due to short cracks. In principle, crack propagation starts from the "initiation" phase (sometimes called "stage I" and mainly being "short crack" propagation) and continues with the "propagation" phase of stage II (where the Paris law is supposed to hold), up to stage III (fast crack propagation) leading to final failure. However, in simple applications, it may be beyond the actual needs to understand the local transient effects on crack propagation unless they have a large effect on the final resulting SN curve. It is clear that if the SN function is of interest, it is redundant to know the details of its derivative, the crack propagation rate curve, also because there is not a single function to be known, nor is the curve in the Paris axis single-valued. Sometimes, the initiation phase is arbitrarily defined with a specific value of final crack size, although it is clear that, depending on the applied stress range, propagation would certainly not start always at the same given size: just thinking of the definition of the long-crack threshold, for example, it is clear that at lower and lower values of stress range, propagation would only start at larger and larger sizes of cracks.

Also, when designing, one set of data that is often available is the SN curve for the base material, as well as the Paris law for long cracks, and thresholds for both fatigue and fatigue crack propagation, as well as ultimate stress and toughness value. One striking difference, often, is the different exponents in the SN curve and in the Paris law, which correspond to the much slower propagation of damage in the uncracked material, as it is reasonable to expect. This suggests that the long-crack propagation rate of the Paris law implies different scaling with respect to number of cycles, and, as we shall see, also the dependence on initial crack size is not the same as that implied by the crack propagation threshold and the toughness value. These differences in turn imply that there must be deviations from these two limit power-law behaviors (the Paris and Erdogan (1963) and the Basquin (1910) one), and one key responsible is certainly cyclic plastic deformation at the crack tip, which in the Paris law regime satisfies "small-scale yielding", i.e., either sufficiently low loads or sufficiently long cracks (Kanninen and Popelar, 1985; Klesnil and Lukas, 1972; Anderson, 1995). These stringent requirements are not always well specified and are also confused, particularly because the Paris law is a beautiful result; but when it does not work, it has the disadvantage of having on the x-axis the stress intensity factor range, i.e., the product of stress range and (square root of) crack size. Hence, if the stress intensity factor range is not the correct parameter to collapse the curve, one is left with the possibility to have a multivalued function. The resulting data points become very sparse and difficult to interpret. For example, "short crack" in particular is nearly always related to the fatigue limit and fatigue threshold (see the definition of intrinsic crack from El Haddad et al. (1979)), whereas the definition of "short" is more correctly framed with respect to the size of the process zone, and hence should also depend on the load level—in the limit of static failure, the equivalent size for the transition from strength-controlled failure to toughness-controlled failure is various orders of magnitude larger than the El Haddad size, so what is "short" in this range is certainly "long" around the fatigue limit region.

Most modifications of Paris' law deal only with single mechanisms of departure from the ideal conditions: threshold limits, both fatigue limit and crack propagation threshold (Laird, 1979; Foreman et al., 1967), crack closure (Elber, 1970), short cracks (Pearson, 1975; Ritchie and Lankford 1986, 1996; El Haddad et al., 1979; Kitagawa and Takahashi, 1976; Kitagawa and Tanaka, 1990), among others. The case of short cracks is one of the most well known since Paris' law can significantly underestimate their rate of growth, and the large number of ad hoc laws reflect the fact that there is not a single type of short-crack deviation. Some authors have suggested a classification of cracks (see Suresh and Ritchie, 1984; Ritchie and Lankford, 1996; Miller, 1999) as follows:

(i) *microscopic short crack* (microstructurally small) for which continuum mechanics breaks down and microstructural fracture mechanics is needed, see for example the models of Hobson et al. (1986) and later Navarro and de los Rios (1988); this is perhaps the most complex category, since crack deceleration or self-arrest is very dependent on the grains size and orientations, and possible decelerations or "minima" in da/dN and multiple small-crack curves can be found (Ritchie and Lankford, 1986);

(ii) *physically small crack* (mechanically small) compared to the scale of local plasticity, for which Elastic–Plastic Fracture Mechanics (EPFM) is needed, first introduced by Tomkins in 1968 (see Miller, 1999) who equated da/dN to crack tip decohesion (from knowledge of the cyclic stress–strain curve), and thence to the bulk plastic strain field that occurs, for example, under high strain fatigue;

(iii) *macroscopic long crack,* growth phase described by Linear Elastic Fracture Mechanics (LEFM).

Clearly, short-crack propagation also occurs below long-crack threshold (Pearson, 1975), but a number of effects tend to modify the crack threshold also, including R-ratio effect, crack closure (in turn due to plasticity-induced or other mechanisms of closure), and not just short crack. Hence, while the detailed understanding of these various effects is still a matter of intense debate, it is remarkable that, as noticed originally by Kitagawa and Takahashi (1976), a quick idea of the threshold for the short cracks can be gathered from the older concept of fatigue limit, and indeed a fitting equation is possible by interpolating between fatigue limit and fatigue threshold (El Haddad et al., 1979). This suggests that modeling of plasticity effects at a short-crack tip can be avoided if one has another asymptotic regime to be matched. This paper, in a sense, tries to extend this kind of reasoning to the crack propagation regime.

In crack propagation studies, a lot of effort has been devoted in recent years to devise accurate models of the near-tip plastic fields (with, possibly, closure as a contact constraint). A simple form is to use cohesive theories of fracture, reminiscent of the Dugdale–Barenblatt strip-yield model for the crack tip plasticity (Dugdale, 1960, Barenblatt, 1962, Budiansky and Hutchinson, 1978; Carpinteri, 1981, 1982), which in principle can be adapted to capture the delicate interplay between bulk cyclic plasticity, closure, and gradual decohesion at the crack tip (Nguyen et al., 2001). Newman (1981, 1992a, b, 1998) and Newman et al. (1999, 2004) further developed such a strip-yield-type model for taking account of plasticity-induced closure, and also to increase the size of the crack by a factor proportional to the (Dugdale–Barenblatt) cyclic yield zone, which is common practise in EPFM to avoid the full use of the J-integral formulation (El Haddad et al., 1980). Initially, the ΔK_{eff}-rate relation is closure-free because these cracks are assumed to be fully open on the first cycle, and small-crack-growth predictions with the crack-closure model agree well, unless there are grain-boundary influences. Perhaps more surprisingly, in the large-crack-growth threshold regime for some materials, the plasticity-induced closure model may not be able to collapse the threshold (ΔK-rate) data onto a unique ΔK_{eff} -rate relation because of other forms of closure, but Paris et al. (1999) say that "Currently, Newman's finite element model is the best we have and worthy of future improvements", since it is argued that it is still able to approximately model most mechanisms of closure. Appropriate "constraint factors" are defined to account that plastic-zone size at a crack front increases as a crack grows from plane-strain conditions at low stress-intensity factors to plane stress as the plastic-zone size becomes large compared to thickness, and loss of constraint is found, associate to the transition from flat-to-slant crack growth (see observations from Schijve, 1967; Newman, 1992a, b).

We shall not try to compete with these models here, if anything, because they involve significant computational effort which we shall try to avoid. Also, in the case of fatigue, strip-yield models need to make significant assumptions since information on cyclic plasticity is needed, whereas present models simply use perfect plasticity, and even if they attempt to include cyclic material properties, enough information is not always known on material constants. A possible alternative approach, explored in this paper, for a unified treatment of long cracks, short cracks (without considering the possible regime of microstructurally small cracks), and fully yielded configurations is an "intermediate asymptotic matching" of the well-known empirical fatigue laws of Wöhler and Paris, obtained by using an increased crack size. To show how to do this, we shall first briefly review such classical laws.

2. Classical laws in fatigue

Initiation and propagation of cracks depend on material, geometry, and load levels (see Suresh, 1998; Fleck et al., 1994). For nominally plain specimen, at low load levels, where we expect fatigue failure at high cycle numbers (HCF), practically the whole life is expended in nucleating the crack (in the sense of stage I short-crack propagation), rather than propagating in stages II and III. At high load levels (those giving low number of

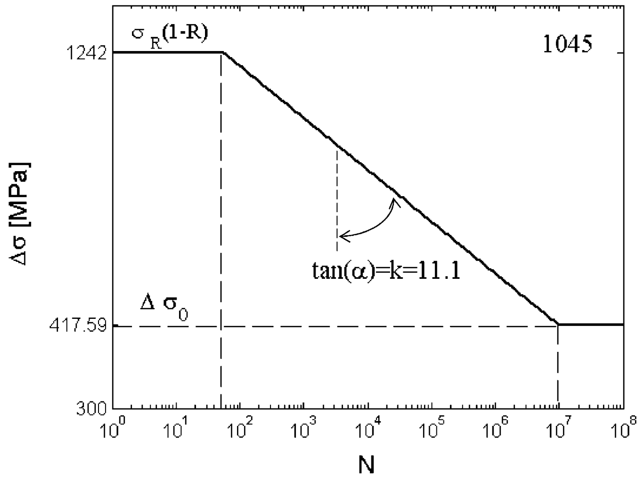

Fig. 1. Schematic Wöhler law (1) for 1045 steel (see example paragraph for the data). Here, $R = -1$.

cycles (LCF)), cyclic plastic deformation takes place rapidly, leading to failure. These various processes result in the well-known empirical Wöhler curve.[1] There is no known fundamental reason to write the curve as a power law, and indeed alternative equations have been suggested, but the power law between two given points is probably the simplest or most used form for the plain specimen (see Fig. 1):

$$N_0 \Delta\sigma_R^k = N_\infty \Delta\sigma_0^k = N_f \Delta\sigma^k = \bar{C}, \quad N_0 < N_f < N_\infty, \tag{1}$$

where $\Delta\sigma_R$ is the range of stress at static failure, $\Delta\sigma_0$ the fatigue limit, and $\Delta\sigma$ the stress range having a life N_f; N_0 and N_∞ are the number of cycles corresponding respectively to $\Delta\sigma_R$ and $\Delta\sigma_0$. Clearly, Eq. (1) also implies

$$k \log F_R = \log \frac{N_\infty}{N_0}, \tag{2}$$

where $F_R = (\Delta\sigma_R/\Delta\sigma_0)$; typically, $N_\infty = 10^7$ and $N_0 = 10^3$, and for steels considering $F_R = 2$, we would have $k = 13.3$, while for $F_R = 3$, $k = 8.4$, in the typical range $k = 6$–14 for Al or ferrous alloys.[2] Notice that this fatigue curve strongly depends on R-ratios, since the static failure can obviously be written as the general yield or ultimate stress condition (i.e. the maximum allowable stress σ_R is equal to σ_y or σ_{UTS}) and since by definition $R\sigma_{\max} = \sigma_{\min}$, $\Delta\sigma_R = \sigma_{\max}(1-R) = \sigma_R(1-R)$. Also, the fatigue limit depends on R-ratio, generally with simple empirical equations as well known from the Goodman or Haigh diagrams (see Suresh, 1998). Hence, both the constants in Eq. (1) are functions of the R-ratio.

[1]Or, in the case of strain-controlled fatigue, in the Basquin–Coffin–Manson's law—but we shall neglect this case for simplicity.

[2]If we have the constants in terms of the Basquin–Coffin–Manson equation, a good approximation is to use the elastic part only at least for the intermediate range of cycles $\Delta\sigma/2 = \sigma_f'(2N')^b$, where σ_f' is called fatigue strength coefficient, b is the fatigue strength exponent, and N' the number of reversals.

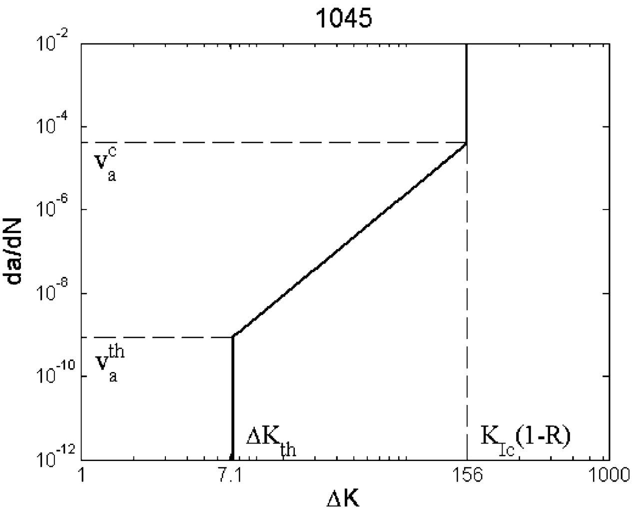

Fig. 2. Schematic Paris law (3) for 1045 steel (see example paragraph). Here, $R = -1$.

Turning to the case of cracked specimen, LEFM applies, and fatigue life (often denominated "residual") is mostly given by stage II propagation, generally by the celebrated Paris' law (Paris and Erdogan, 1963). The Paris law gives the advancement da of fatigue crack per unit cycle dN, v_a, as a function of the amplitude of stress intensity factor ΔK (see Fig. 2):

$$v_a = \frac{\mathrm{d}a}{\mathrm{d}N} = C\Delta K^m, \quad \Delta K_{\mathrm{th}} < \Delta K < K_{Ic}, \tag{3}$$

where ΔK_{th} is the "fatigue threshold" and K_{Ic} the "fracture toughness" of the material (assuming $R = 0$); C and m are the so-called Paris' constants. There is therefore no dependence on absolute dimension of the crack. The law is mostly valid in the range 10^{-8}–10^{-6} mm/cycle, intersecting ΔK_{th} and K_{Ic} at $v_a^{th} = 10^{-9}$ mm/cycle, and $v_a^c = 10^{-5}$ mm/cycle, respectively, where v_a^{th} is a conventional velocity at the threshold, and v_a^{c} at the critical conditions (see Fig. 2). This means that the constant C is not really arbitrary, since by writing the condition at the intersections, $C = v_a^{\mathrm{th}}/\Delta K_{\mathrm{th}}^m = v_a^{\mathrm{c}}/K_{Ic}^m$. In practice, these limits are not precise and only standards can help defining reference values.

From the linearity in the range 10^{-9}–10^{-5} mm/cycle in the log/log plot, Fleck et al. (1994) suggest to find the Paris exponent m as

$$\mathrm{Log}\, F_K = \frac{4}{m}, \tag{4}$$

where $F_K = (K_{Ic}/\Delta K_{\mathrm{th}})$, and their paper (see specifically Fig. 16) seems to confirm this assumption for the exponent m. More generally, the following relation holds:

$$m\, \mathrm{Log}\, F_K = \mathrm{Log} \frac{v_a^{\mathrm{c}}}{v_a^{\mathrm{th}}}. \tag{5}$$

An obvious link between the two curves (Wöhler and Paris) is obtained when considering the life of a distinctly cracked specimen having an initial crack size a_{i}. Under the assumptions of constant remote stress and no geometrical effects, for $m > 2$, the following is obtained (where the dependence on the final size of the crack a_f has been removed as relatively not influential in most cases):

$$a_i^{(2-m/2)} = \left(\frac{m-2}{2}\right) C\pi^{m/2}\Delta\sigma^m N_f. \tag{6}$$

This is to be considered as a Wöhler curve of the cracked component, and the Wöhler equivalent exponent in these conditions, k', to distinguish it from the "material constant" base value k, turns out to be exactly equal to the Paris exponent, $k' = m$. It is interesting, however, to remark that the SN fatigue curve depends on the initial crack size, a_i. Hence, the threshold condition from Eq. (6) would tend not to coincide with that directly obtained from the threshold value, which also depends on a_i but with a different power:

$$\Delta\sigma_{\mathrm{lim,th}} = \frac{\Delta K_{\mathrm{th}}}{\sqrt{\pi a_i}}. \tag{7}$$

The two powers in Eqs. (6) and (7) coincide only if $(m-2)/(2m) = 1/2$, which is only true asymptotically for very high m, showing in fact that the Paris law should near the threshold have some near-threshold deviation, as suggested already by Donahue et al. (1972) as (see Fig. 3)

$$\mathrm{d}a/\mathrm{d}N = A(\Delta K - \Delta K_{\mathrm{th}})^m. \tag{8}$$

Notice that the threshold strongly depends on the R-ratio and this is a matter of further empirical equations that are omitted here for simplicity. For stage II, Paris' law in Eq. (3) works correctly, up to deviations in region III, which in turn depend strongly on the R-ratio since the fracture mode becomes essentially static and governed by the maximum value of the stress intensity factor: the toughness condition for $K_{\max} = K_{Ic}$ need to be involved, requiring $RK_{\max} = K_{\min}$, and hence $\Delta K = K_{\max}(1-R) = K_{Ic}(1-R)$. The curve exhibits a rapidly increasing growth towards ductile tearing and/or brittle fracture, and a possible transition curve was proposed first by Foreman et al. (1967):

$$\mathrm{d}a/\mathrm{d}N = B\Delta K^m/[(1-R)K_{Ic} - \Delta K]. \tag{9}$$

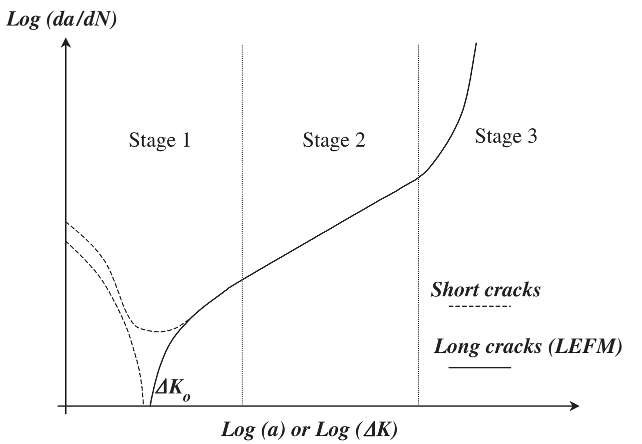

Fig. 3. A schematic of the typical fatigue growth behavior of cracks.

Combinations of high and low ΔK values also exist (Kanninen and Popelar, 1985; Carpinteri et al., 1994), a good example being the one proposed by McEvily and Groeger (1977):

$$da/dN = D(\Delta K - \Delta K_{\text{th}})^2[1 + \Delta K/(K_{Ic} - K_{\max})]. \tag{10}$$

Notice the power law here is 2 and not general m as in Eq. (5), but this could be easily generalized. Notice also that these laws show the R-ratio effect only near the K_{Ic} limit, but since the threshold depends on R also, this is implicitly defined. It is clear that the beauty of Paris' law also concerns the very mild dependence on R-ratio in stage II.

3. Length scales in fatigue

So far, we have only dealt with the case of either completely uncracked or the distinctly cracked specimen, for which the Wöhler and Paris laws apply. But where does the transition occur? A qualitative idea is immediately found from a simple dimensional analysis in the case of infinite life or static failure. In fact, LEFM has introduced two new material constants (fracture toughness in static failure, fatigue threshold in fatigue) for the case of cracked specimen, which differ from the more classical corresponding material properties (ultimate strength and fatigue limit) from a square root of a length scale: hence at fatigue limit, the transitional size is the constant a_0 defined for a given threshold stress intensity range ΔK_{th} and fatigue limit $\Delta\sigma_0$ as

$$a_0 = \frac{1}{\pi}\left(\frac{\Delta K_{\text{th}}}{\Delta\sigma_0}\right)^2. \tag{11}$$

For the interaction between fatigue limit and fatigue threshold, usually a "matching asymptotics" empirical equation is used (El Haddad et al., 1979) for the generic stress range fatigue limit, $\Delta\sigma_f$:

$$\Delta\sigma_f = \Delta K_{\text{th}}/\sqrt{\pi(a + a_0)}, \tag{12}$$

which suggests the perhaps misleading definition of a_0 as the "intrinsic crack". Naturally, the corresponding static failure case introduces the dimensions a_0^S analogous to that defined in Eq. (11), and depending here on K_{Ic}, the toughness of the material and σ_R its tensile strength (either ultimate stress UTS or yield stress) as

$$a_0^S = \frac{1}{\pi}\left(\frac{K_{Ic}}{\sigma_R}\right)^2. \tag{13}$$

In general,

$$F_K = \left(\frac{K_{Ic}}{\Delta K_{\text{th}}}\right), F_R = \left(\frac{\Delta\sigma_R}{\Delta\sigma_0}\right),$$

and for metals, typical values are 5–20 and 2, respectively, so that a_0^S is about 100 times greater than a_0.

In order to apply Paris' law, the assumptions of LEFM need to be fulfilled. This certainly means "long crack" behavior, as discussed in the introduction, and this requires $a > a_0$, as defined in Eq. (11), or may be higher for higher stress ranges, i.e., shorter lives, up to the limit case of static failure, for which the appropriate condition is rather $a > a_0^S$, as defined in Eq. (13). Short cracks see significantly larger crack propagation rates, as also discussed earlier. The dotted lines in the schematic diagram of Fig. 3 illustrate some of the possible short-crack behavior.

Here, an approach is derived for the intermediate unified law in fatigue: considering that the "intrinsic crack" size, as in the El Haddad equation, should increase from the fatigue limit size, a_0, towards the a_0^S, we shall assume that this size can be used in a recent generalization of the stress intensity factor (Pugno, 2004; Pugno and Ruoff, 2004; Taylor et al., 2005), and its dependence with stress range shall be assumed such that the correct trend towards Wöhler's law is obtained.

4. Generalized Paris' law

Quantized Fracture Mechanics (QFM) (Pugno and Ruoff, 2004) can generalize Paris' equation, by substituting $K(a)$ with an appropriate mean value, $K^*(a,\Delta a)=\sqrt{\langle K^2(a)\rangle_a^{a+\Delta a}}$, where Δa is "fracture quantum", a material/structural material constant, which for infinite life is clearly related to the a_0 in Eq. (11) and for static failure to a_0^S of Eq. (13). In order to consider the effect of this "apparent" additional crack size, it is natural in the study of fatigue crack growth to propose the following generalized Paris' law (Pugno, 2004):

$$\frac{\mathrm{d}a}{\mathrm{d}N}=C(\Delta K^*(a,\Delta a,\Delta\sigma))^m, \tag{14}$$

where in turn Δa is a function of $\Delta\sigma$, since in proximity to the fatigue limit, it would be a_0 as in Eq. (11), and in proximity of static failure, it would be a_0^S of Eq. (13). By integrating Eq. (14), the total number of cycles N_C^{P*} can be found for the fatigue collapse, arising when the crack length has reached its critical final value a_C:

$$N_C^{P*}=\frac{1}{C}\int_a^{a_C}\frac{\mathrm{d}a}{(\Delta K^*(a,\Delta a,\Delta\sigma))^m}. \tag{15}$$

Accordingly, in the criterion of Eq. (14), we can fix Δa to recover, in the limit case of $a\to 0$, Wöhler's prediction, Eq. (1), which we write here as

$$N_C^W=\frac{\bar{C}}{\Delta\sigma^k}. \tag{16}$$

Hence,

$$\Delta a : N_C^{P*}(a\to 0)=N_C^W. \tag{17}$$

Thus, Eqs. (14) or (15), with the position of Eq. (17), can be considered a generalized Paris' law. Note that such a law is of very simple application, and would allow one to study not only the final condition but also the evolution of the fatigue crack growth $N^{P*}(a(N))$, where $a\leqslant a(N)\leqslant a_C$.

5. The example of the Griffith crack

As a simple example of application, let us consider the Griffith case (infinite elastic plate with a symmetric crack of length $2a$). For this case, the stress intensity factor (mode I) is

$$K=\sigma\sqrt{\pi a}, \tag{18}$$

Accordingly, by integration,

$$K^* = \sigma\sqrt{\pi(a + (\Delta a/2))}. \tag{19}$$

By applying Eq. (14) and integrating between an initial and a final value of crack size (we suppose $a_f = a_c$ and $a_i = a$), it follows:

$$N_C^{P*} = \frac{1}{C\Delta\sigma^m\pi^{m/2}}\frac{(a_C + (\Delta a/2))^{1-m/2} - (a + (\Delta a/2))^{1-m/2}}{1 - m/2}. \tag{20}$$

From Eq. (17), Δa can be obtained by solving

$$\frac{1}{C\Delta\sigma^m\pi^{m/2}}\frac{(a_C + (\Delta a/2))^{1-m/2} - (\Delta a/2)^{1-m/2}}{1 - m/2} = \frac{\bar{C}}{\Delta\sigma^k}. \tag{21}$$

Assuming $a_C \gg \Delta a$, we obtain

$$\Delta a \approx 2\left(a_C^{1-m/2} - \frac{C\bar{C}\pi^{m/2}(1 - m/2)}{\Delta\sigma^{k-m}}\right)^{1/(1-m/2)}. \tag{22}$$

For $m>2$ (usual case), $-2<1/(1-m/2)<0$, the first term can be neglected (a_C being a large crack–see below). Hence it is convenient to rewrite Eqs. (21) and (22) as

$$N_C^{P*} \approx \frac{1}{C\Delta\sigma^m\pi^{m/2}}\frac{\left(a + (\Delta a/2)\right)^{1-m/2}}{m/2 - 1}, \tag{23}$$

$$\Delta a \approx 2\left(\frac{\Delta\sigma^{k-m}}{C\bar{C}\pi^{m/2}\left(m/2 - 1\right)}\right)^{1/(m/2-1)}, \tag{24}$$

i.e., a power law of stress range with exponent $(k–m)/(m/2–1)$. For example, typical values for a metal $m = 4$, $k = 12$, then $(12–4)/(2–1) = 8$. Hence, Δa increases remarkably fast with the stress range, similarly to what was found from simpler independent reasoning (Ciavarella, 2002).

By assembling Eqs. (23) and (24), one obtains

$$N_C^{P*} \approx \frac{1}{C\Delta\sigma^m\pi^{m/2}}\frac{\left(a + \left(\Delta\sigma^{k-m}/(C\bar{C}\pi^{m/2}(m/2 - 1))\right)^{1/(m/2-1)}\right)^{1-m/2}}{m/2 - 1}, \tag{25}$$

which is the proposed integrated Paris law. Notice that obviously it is not a power law type, but a more complicated law depending on initial size of the crack and stress range. An obvious limit case is for very large crack, for which one obtains from Eq. (25)

$$N_C^{P*} \approx \frac{1}{C\Delta\sigma^m\pi^{m/2}}\frac{(a)^{1-m/2}}{m/2 - 1}, \tag{26}$$

which is also the integrated form of the original Paris law, Eq. (6), which can be recast in the form. Notice that "large crack" can be made quantitative, indicating that

$$a \gg \left(\frac{\Delta\sigma^{k-m}}{C\bar{C}\pi^{m/2}\left(m/2 - 1\right)}\right)^{1/(m/2-1)}. \tag{27}$$

Conversely, "short crack" for "$\ll$" and returning to the typical cases of metals $2<m<k$, short crack is obtained if either the numerator (stress range) is large and/or the denominator is small. Vice versa, the usual definition of short crack is (as discussed earlier) limited to the absolute size, probably because the stress range at which this is measured is not too far from the fatigue limit (or just above it).

Eq. (25) is a "non-power-law", which we can consider as an "asymptotic matching" between the two power-law regimes (Wöhler and Paris) at the extremes. In fact, Eq. (25) in the other limit of small a becomes:

$$N_C^{P*} \approx \frac{1}{C\Delta\sigma^m \pi^{m/2}} \frac{\left(\Delta\sigma^{k-m}/(C\bar{C}\pi^{m/2}(m/2-1))\right)^{-1}}{m/2-1} = N_C^W, \tag{28}$$

which is the original Wöhler law, Eq. (16). Notice that since we impose to obtain the Wöhler law for negligible crack, we are not sure what the reduction of the fatigue limit is for the a_0 crack size (this will be discussed further in the example case paragraph).

This result can also be interpreted in terms of the generalized Paris law, which reads for a Griffith crack:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\left(\Delta\sigma\sqrt{\pi\left(a + \left(\frac{\Delta\sigma^{k-m}}{C\bar{C}\pi^{m/2}\left(m/2-1\right)}\right)^{1/(m/2-1)}\right)}\right)^m. \tag{29}$$

For short cracks ($a \to 0$), one gets

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\pi^{m/2}\left(C\bar{C}\pi^{m/2}(m/2-1)\right)^{(-m/2)/(m/2-1)} \Delta\sigma^{(k-m)((m/2)/(m/2-1))+m}, \tag{30}$$

i.e., a Paris power law in terms of $\Delta\sigma$ rather than ΔK.

6. An example

To make an illustrative example of a metallic material with a given Paris equation and Basquin law, the SAE1045 steel is considered (see Table 1).

The fatigue limit (at 10^7 cycles) is from the Basquin law $\Delta\sigma_0 = 418$ MPa; also, $k = 11.11$, $\bar{C} = 1.32\,E36$ MPa^{12}. The crack size a_0 is 92 µm, whereas static size a_0^S is 5.3 mm. In Figs. 4 and 5, the predictions of Eqs. (25) and (29) for $a/a_0 = 1$, 10, 100, 1000 are compared, respectively, with the classical Wöhler and Paris laws.

Table 1
Mechanical properties of SAE1045 steel[a]

Material	Stress units	UTS	Yield strength	Fatigue strength coeff.	Fatigue strength exponent	Crack growth coeff.	CGC units/ cycle	Crack growth exponent	Thresh SIF $R=0$	Fracture toughness
1045	MPa	621	382	948	–0.09	8.20E–13	m	3.5	7.1	80

[a]Data from Multiaxial Fatigue, Analysis & Experiments, Eds. G.E. Leese, D. Socie, SAE Pub. AE-14, 1989, for SAE1045 steel.

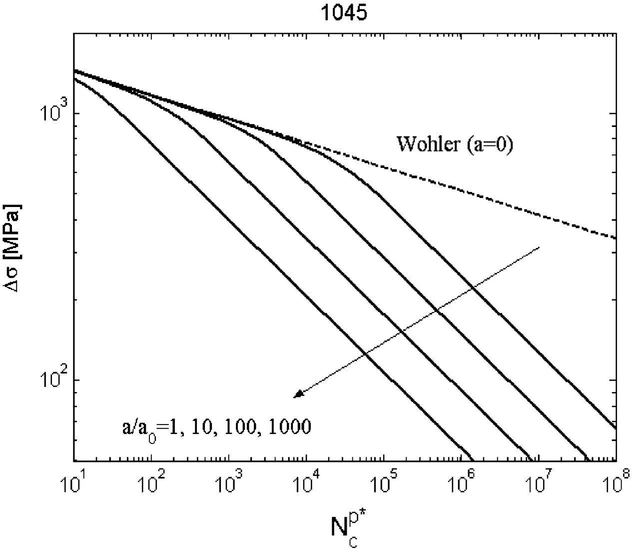

Fig. 4. SN curves according to Eq. (29).

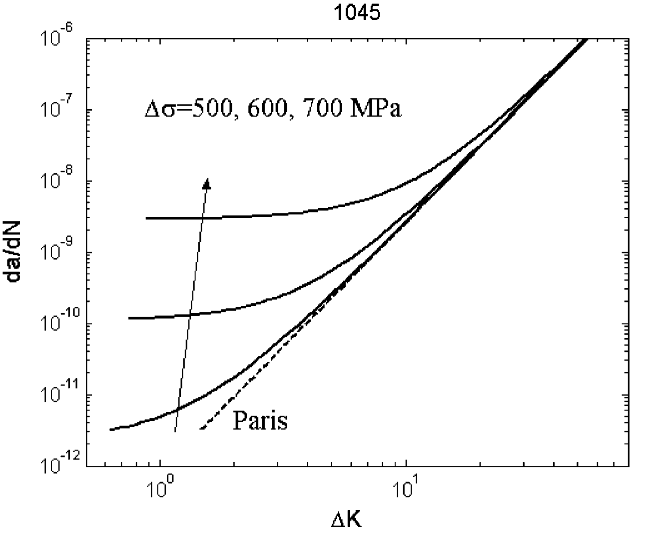

Fig. 5. Crack propagation laws from Eq. (25) with fixed $\Delta\sigma$ (in MPa in the legend). Notice that with fixed $\Delta\sigma$ we obtain substantially larger propagation rates. ΔK is in MPa $m^{1/2}$ and $\mathrm{d}a/\mathrm{d}N$ is in m/cycle.

Notice we can rewrite Eq. (29) explicitly in terms of stress intensity factor range as in this form, it is clear that we can either fix $\Delta\sigma$ (and Fig. 5 is obtained) which returns to Paris for low stress ranges, or fix a (not shown) which gives Paris for large a. At very small crack sizes, in other words, our application of Paris' law is rather an application of Wöhler's law as found already in Eq. (30) and in this example case reads

$$\frac{\mathrm{d}a}{\mathrm{d}N} = 0.93 \times 10^{-69} \Delta\sigma^{21.3}. \tag{31}$$

For $a = 0$, the standard Wöhler law is obtained, and for progressively larger a, a significant transition towards a Paris regime deviation occurs (particularly at high number of cycles, i.e., low stress range levels $\Delta\sigma$, in MPa). Notice that we have not included here the effect of the crack propagation threshold, which should truncate the Paris curves.

Turning to Fig. 5, we remark that only for high $\Delta\sigma$ we obtain deviations from the standard Paris law, and in this form, the law seems to follow a horizontal asymptote. Here in fact, to obtain a given level of ΔK, for a constant $\Delta\sigma$, we need to increase the size of the crack, and hence eventually we get closer to the Paris law. Clearly, for higher and higher $\Delta\sigma$, we need larger and larger crack sizes to reobtain the validity or LEFM and of the standard Paris law in particular.

7. Some comparisons with existing approaches

The most obvious comparison of our approach would be that with the strip-yield models (Newman 1981, 1992a, b, Newman et al., 2004, Nguyen et al., 2001), and others. Naturally, we do not use these models for predicting crack closure, but only for a "virtual" increase of the size of the crack, due to process zone effects, interpreted in terms of their resulting effects in the SN curve of the uncracked material. These models are very much under development, and it would be almost impossible to make a detailed comparison, without referring to specific cases, and detailed computational effort. Naturally, our law cannot pretend to be extremely accurate in terms of crack propagation rate, although it is also equally possible that in some particular case, when the detailed yield-strip models are not correctly "calibrated", they can extrapolate results incorrectly, whereas our models should maintain the advantage of being an "interpolation" model.

The laws obtained, particularly for short cracks, would seem to suggest that the crack propagation rate of a short crack should be proportional to some power law of the stress range. A similar conclusion was obtained in relatively not well-known papers by Nisitani (1981) (see also Nisitani and Goto, 1987, and finally Nisitani et al., 1992) and Murakami et al. (1983). In particular, Murakami et al. (1983) and Murakami and Miller (2005) more recently conduct cyclic-stress-amplitude-controlled fatigue tests with a material (S45C medium carbon steel) showing a stable cyclic stress–strain curve, so that these can also be considered approximately as cyclic-strain-amplitude-controlled fatigue tests. They conclude with a short-crack propagation law of the type

$$\mathrm{d}a/\mathrm{d}N = H\Delta^h a, \tag{32}$$

where H is a constant and $h \approx 8$. This law is remarkably close to our result (Eq. (30)), with the exception of the further dependence on the crack size. Notice that the integration between initial and final size of crack a_i, a_f, for the cycles from $N_i \ll N_f$, leads to the logarithmic dependence on the crack size as

$$\log(a_\mathrm{f}/a_\mathrm{i}) = H\Delta\sigma^h N_f, \tag{33}$$

which, with respect to standard Basquin-type laws, clearly evidences a very mild dependence on initial crack size. This is compatible if we assume that dependence on the initial crack size could easily have escaped the attention of previous researchers, who, however, have noticed almost invariably a large amount of scatter in standard SN curves. This scatter is in turn possibly interpreted with Eq. (33) in the light of the fact that the initial crack will have a random size. Nisitani et al. (1992) also give a number of other details and suggested mechanisms and equations. In particular, a unified explanation for two growth laws (Paris and short crack) is made based on an assumption that *the crack growth rate is proportional to the reversible plastic-zone size*, both on the surface and in the interior, regardless of the scale of yielding. Moreover, reduced forms of small-crack-growth law are given in which the effect of mechanical properties is partly considered from the ultimate strength of the material only. A partial account is taken of R-ratio, suggesting only the coefficients in the laws would change.

Finally, the papers by Polák (2005a, b) deserve some special attention, which contain similar proposals, despite the connection made here between the Manson law on plastic strain amplitude dependence and how this can be interpreted in terms of short-crack growth.

8. Discussion

As we have seen, various attempts are emerging to unify the classical fatigue laws (Basquin and the Coffin–Manson equations) with the crack propagation laws (Paris and its deviations). In general, if the fatigue crack grows from an initial defect size a_{i} to a final size a_{f} according to the general expression

$$\mathrm{d}a/\mathrm{d}N = B\Delta\sigma^n a^m, \tag{34}$$

where B, m, and n are material constants, then upon integration of Eq. (34) between the limits a_{i} and a_{f}, both the Basquin and the Coffin–Manson equations can be derived. This is a qualitative connection between the two theories, where, however, the exponents change in the two limits. The main difficulty with the stress and strain range approaches is that no account is taken of the behavior of cracks; hence, the extent of fatigue damage between the start and the end of the fatigue process cannot be determined, and they tend to work only when crack propagation is a very minor part of life. Also, it has been shown that if the differential equation describing the crack growth process has separate variables, then the Palmgren–Miner damage accumulation rule is true (Svensson and Mare, 1999; Todinov, 2001). Obviously, Eq. (34) satisfies this condition, and hence it does not take account of the load sequence effect, which could be significant in some situations. Hence, there is a connection between deviations from the Paris law regime (or Basquin–Coffin–Manson regime) and deviations from the Palmgren–Miner rule. There is still a lot to be understood before a crack growth approach becomes accurate enough for fatigue life prediction. Today, it is not commonly used for fatigue design in industry, mainly because of two main difficulties: (i) the initial crack size a_{i} is often unknown and (ii) the data of $\mathrm{d}a/\mathrm{d}N$ vs. ΔK are more expensive to obtain. Our approach tries to avoid the requirement for an initial crack length a_{i}, by interpolation of the two approaches. A secondary implication of our result is that a large scatter of fatigue lives is probably intrinsic in the uncracked specimen case. This indicates that the accuracy required for practical complex structures under more complex loading is higher than that for plain specimens.

Another difficulty is that these fatigue laws remain essentially empirical, and hence their constants often vary more than expected. For example, various attempts to find more fundamental reasons for the Paris law to be applied (other than the self-similarity of the propagation process, which would seem to suggest a fixed coefficient m of either 2 or 4 depending on the chosen mechanism) have failed to be generally convincing (see Fleck et al., 1994), and indeed, as discussed by Barenblatt and Botvina (1980), the effect of other parameters on m is expected, including the thickness of the plate, via an additional parameter, related to the ratio of specimen thickness to cyclic plastic region size. Also, as remarked in the recent paper by Spagnoli (2005), fractality of the crack surface may explain power-law dependences on the absolute crack size (size-effects) of Paris' coefficient C for certain quasi-brittle materials. It is not possible here to review all the various proposals for the extension of Paris' law, which would take account of the many experimentally observed deviations.

Finally, a remark is made on the influence of the $R = \sigma_{\min}/\sigma_{\max}$ ratio on fatigue crack growth rate $\mathrm{d}a/\mathrm{d}N$ (e.g., see Fig. 2). As described in paragraph 2, there are various extensions of the classical Paris law for including such a ratio near the threshold region and the stage III region. However, these effects are correctly modeled in these equations only for long cracks. For short cracks (and hence we refer particularly to the near the threshold region), the effects of R tend to be different, and, as described in the introduction, require detailed account of crack closure and its various mechanisms, still very much debated by the scientific community. However, our approach is very direct and perhaps explains these deviations in a more straightforward way, since the mean stress effects are better known in the context of Wöhler's law constants than in the Paris' law constants for short cracks. The fact that mean stress effects in the Wöhler law have been dealt for many years with empirical laws studying, for example, the effect of mean stress on fatigue limit by means of diagrams such as Goodman or Haigh (see Suresh, 1998), and the resulting linear or parabolic decays with mean stress (Goodman line and Gerber parabola, respectively), opens the way for further results in the area of short cracks (at least in terms of "baseline" results), with extensions of the present approach. This, however, is outside the scope of the present paper, and will be left for further investigations.

9. Conclusions

We have proposed a new equation generalizing Paris' law for fatigue crack propagation by using the ideas of adding a fracture "quantum" to the standard LEFM, which has recently been applied successfully to the case of static failure and to the case of infinite life (Taylor et al., 2005). This paper attempts the further generalization as a more general interpolation procedure, not only between static failure and infinite life (i.e., finite life) but also between the two regimes of strength-controlled failure and stress-intensity-factor-controlled failure. Because the interpolation procedure essentially works on the integrated form of the crack propagation law, i.e., in terms of SN curve, it cannot be expected to lead to other than qualitative (or averaged) results for the instantaneous crack propagation rate. By imposing consistency with Wöhler's law for the uncracked material, in the limit when the new generalized law is used for short initial cracks, we get the appropriate "quantum crack" size. The fracture "quantum" with applied stress range, in the limit of very large cracks and static failure, is expected to be very close to a plastic radius, whereas for conditions close to the fatigue limit and threshold should be close to the El Haddad intrinsic crack size a_0.

In these respects, the proposed model has the advantage of being an "interpolation procedure" between the celebrated Paris and Wöhler regimes (or perhaps, more elegantly, an "intermediate asymptotics" solution in Barenblatt's sense), and hence avoids the risk associated with the inevitable "extrapolation" nature of the many other phenomenological but essentially empirical models. These two limits correctly modeled suggest a qualitative experimental verification, but the model, in the present form, remains essentially speculative.

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