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Towards Chaos in Vibrating Damaged Structures—Part II: Parametrical Investigation

The aim of the present paper is to evaluate the complex oscillatory behavior, i.e., the transition to chaos, in damaged nonlinear structures under excitation. In the present paper, Part II, we apply the theoretical approach described in Part I to perform an extensive parametrical investigation. We focus our attention on a cantilevered beam with several breathing cracks subjected to sinusoidal excitation. The numerical simulations have been performed by varying the number of cracks, their depth and position, as well as the amplitude, frequency and position of the excitation, for a total of 83 different cases. [DOI: 10.1115/1.1934631]

1 Introduction

As shown in Part I, the proposed theoretical and numerical approach can be successfully applied to the study of damaged structures. The aim of the present paper, Part II, is to perform an extensive parametrical investigation to describe the influence of the main parameters on the dynamic behavior of the considered system. For a given model, the system complexity is a function of the complexity of the structure, as well as of the complexity of the excitation. Focusing our attention to the excited cracked cantilevered beam introduced in Part I, we have performed a parametrical investigation by varying the main parameters influencing the structural complexity, i.e., the cracks' number, depth and position, as well as the force amplitude, frequency and position, for a total of 83 different cases.

Several researchers have studied the problem of a beam with a breathing crack from analytical, numerical and experimental viewpoints [1–7]. In particular, relevant numerical investigations have been presented in Ref. [1], by using the Finite Element Method and in Ref. [2], applying directly numerical integration. In spite of this, an extensive parametrical investigation on the topic is entirely absent in the literature and is the object of the present paper (Part II).

The method, described in detail in Part I, has permitted to capture the influence of the different parameters on the complex behavior for the nonlinear structure, as well as the transition towards deterministic chaos, i.e., towards a nonperiodic response of the structure subjected to periodic excitation.

In particular, we have found that, if a weak nonlinearity is considered, only offset and super-harmonic components can be observed in the structural response. On the other hand, if the nonlinearity becomes stronger, also sub-harmonic components can be observed in the structural response, providing the so-called *complex* behavior.

Furthermore, the influence of each parameter on the structural behavior will be discussed on the basis of the presented extensive parametrical investigation. A new methodology for vibration-based inspections will also be presented.

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2 Parametrical Simulations

As an example, we focus our attention onto a clamped beam. It is 270 mm long and has a transversal rectangular cross section of base and high, respectively, equal to 13 and 5 mm. The material is UHMW-ethylene, with a Young's modulus of 8.61×10^8 N/m^2 and a density of 935 kg/m^3. We have assumed a modal damping of 0.002. The beam has been discretized with 20 finite elements. We have found that a Complexity Index $\Theta=4$ and a number of terms $N=16$ give a good approximation (i.e., for larger values of Θ and N substantially identical solutions are obtained). The first natural frequency of the undamaged structure is $f_u=10.6$ Hz.

The extensive parametrical investigation has been performed by varying the main parameters quoted in Fig. 1. These parameters affect the behavior of the system, as summarized in the following:

(A) By varying the depth of a crack localized at one-half of the total length of the beam;
(B) by varying the depth of a crack localized at one-third of the total length of the beam;
(C) by varying the crack position;
(D) by varying the excitation frequency;
(E) by varying the excitation amplitude;
(F) by varying the depth of one crack (in a beam containing two cracks);
(G) by varying the position of the excitation (in a beam containing two cracks).

Each of these families of parametrical simulations is separately treated in a specific section. The outputs from each simulation are the same as presented in the examples of Part I. As structural response we present only the normalized amplitude, corresponding to a given frequency component. It is defined as the ratio of the amplitude of the considered frequency component to the amplitude of the linear one (the component of the response with the same frequency of the excitation) related to the displacement of the free-end, i.e.,

$$\text{Normalized Amplitude}\Big|_{j\neq\Theta}=\frac{\sqrt{A_{20j}^2+B_{20j}^2}}{\sqrt{A_{20\Theta}^2+B_{20\Theta}^2}} \qquad (1)$$

with reference to the variables introduced in Part I.

2.1 Parametrical Simulations by Varying the Depth of a Crack Localized at One-Half of the Total Length of the Beam (A). These numerical simulations consider one crack with a variable depth of a_1. They are indicated by the letter A. Referring to Fig. 1, the coordinate of the crack is $d_1=135$ mm (at one-half of the total length of the beam), the force amplitude is $F=5$N with a

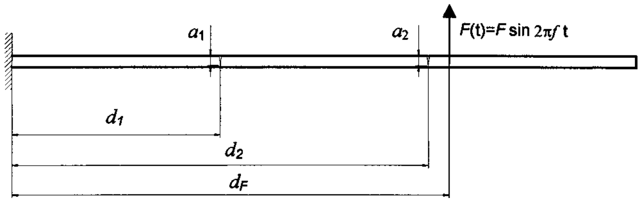

Fig. 1 The considered nonlinear system and the main parameters numerically investigated

Table 1 One crack (localized at one-half of the total length of the beam)—Numerical simulations by varying the crack depth (A)

One crack—a_1 variable d_1=135 mm; F=5N; f=25 Hz	
a_1 [mm]	*Case*
1.0	A1
2.0	A2
2.2	A3
2.4	A4
2.6	A5
2.8	A6
3.0	A7
3.2	A8

frequency f of 25 Hz (to compare with the first natural frequency of the undamaged structure, f_u=10.6 Hz). The parameters of the simulations are summarized in Table 1. The numerical responses in terms of the normalized amplitude of Eq. (1) as a function of the crack depth are presented in Fig. 2. Figure 2(a) considers the frequency components $j=0,1,2,3$, as well as Figs. 2(b)–2(d), respectively, the components groups $j=5-8; j=9-12-j=13-16$. It is very interesting to note that the first symptom of the presence of a crack, i.e., of the nonlinearity, is the offset (j=0, 0-frequency) in Fig. 2(a), as well as the super-harmonic components, i.e., $j=8(2\omega)$, $j=12(3\omega)$, and $j=16(4\omega)$, which are also present for small crack depths. As a consequence, we can affirm that the nonlinearity implies a natural rupture of the symmetry of the problem (i.e., an offset). A rather considerable presence of sub-harmonic components arises after a threshold value of crack depth, which is around one-half of the total height of the beam. For this excitation frequency (around twice the first natural frequency), the component of *period doubling* ($\omega/2$) and its multiples ($3\omega/2,\ 5\omega/2,\ 7\omega/2$) are clearly prevailing.

2.2 Parametrical Simulations by Varying the Depth of a Crack Localized at One-Third of the Total Length of the Beam (B). These numerical simulations consider one crack with a variable depth of a_1. They are indicated by the letter B. Referring to Fig. 1, the coordinate of the crack is d_1=90 mm (at one-third of the total length of the beam), the force amplitude is F=2N with a frequency f of 19 Hz. The parameters of the simulations are summarized in Table 2. The numerical responses in terms of the normalized amplitude of Eq. (1) as a function of the crack depth, are presented in Fig. 3. The results are similar to those of the previous

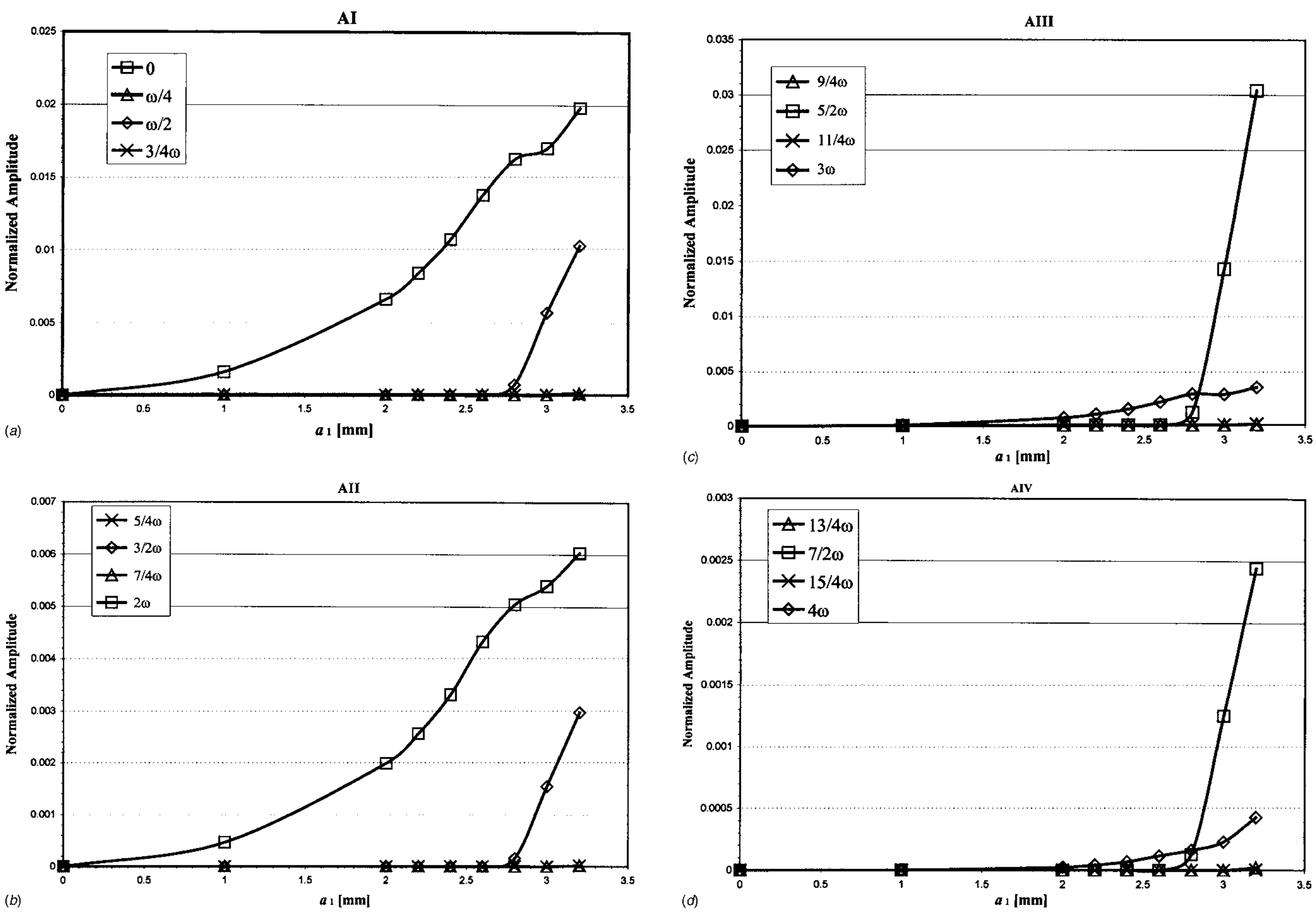

Fig. 2 One crack (localized at one-half of the total length of the beam)—Numerical simulations by varying the crack depth (*a*) (AI). (*b*) (AII). (*c*) (AIII). (*d*) (AIV).

Table 2 One crack (localized at one-third of the total length of the beam)—Numerical simulations by varying the crack depth (B)

One crack—a_1 variable $d_1=90$ mm; $F=2$N; $f=19$ Hz	
a_1 [mm]	*Case*
1.0	B1
2.0	B2
2.2	B3
2.4	B4
2.6	B5
2.8	B6
3.0	B7
3.2	B8
3.4	B9
3.6	B10
3.8	B11
4.0	B12
4.2	B13
4.4	B14
4.6	B15
4.8	B16

Table 3 One crack—Numerical simulations by varying the crack position (C)

One crack—d_1 variable $a_1=4.25$ mm; $F=5$N; $f=25$ Hz	
d_1 [mm]	*Case*
260	C1
240	C2
220	C3
200	C4
180	C5
160	C6
140	C7
120	C8
100	C9
80	C10
60	C11
40	C12

case. The main difference herein is that we have a higher nonlinearity due to the reduction of the distance between crack and clamp. In addition, in this case we have also considered a crack with a higher depth. If the nonlinearity increases (larger crack depth or lower distance between crack and clamp), other subharmonic components, not necessarily a multiple of that of period doubling, arise in the dynamic response. In this case, as in the previous one, the frequency of the excitation is around twice the first natural one.

2.3 Parametrical Simulations by Varying the Crack Position (C). These numerical simulations consider one crack with a depth of $a_1=4.25$ mm. They are indicated by the letter C. Referring to Fig. 1, the coordinate d_1 of the crack is assumed variable; the force amplitude is $F=5$N with a frequency f of 25 Hz. The pa-

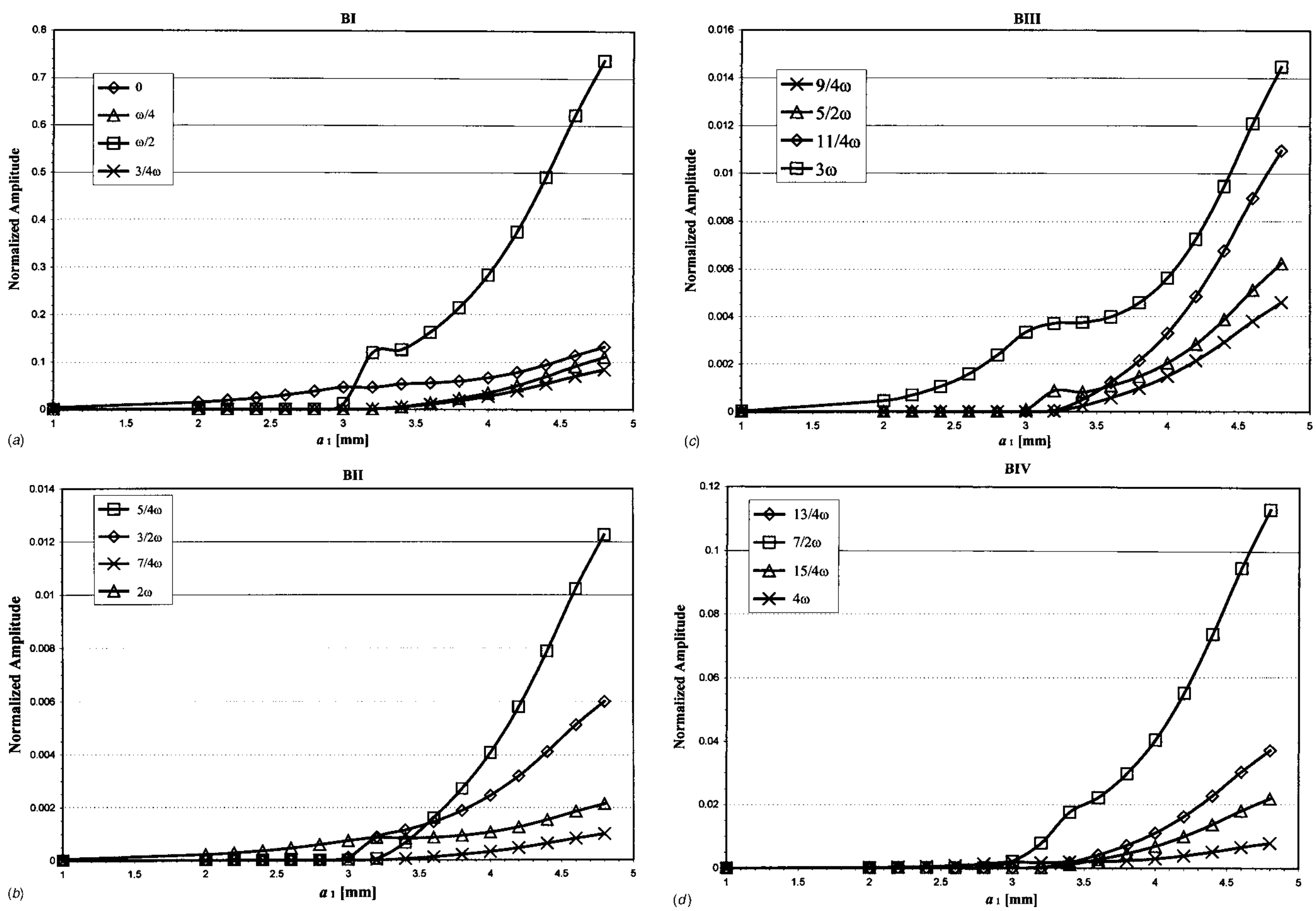

Fig. 3 One crack (localized at one-third of the total length of the beam)—Numerical simulations by varying the crack depth (*a*) (BI). (*b*) (BII). (*c*) (BIII). (*d*) (BIV).

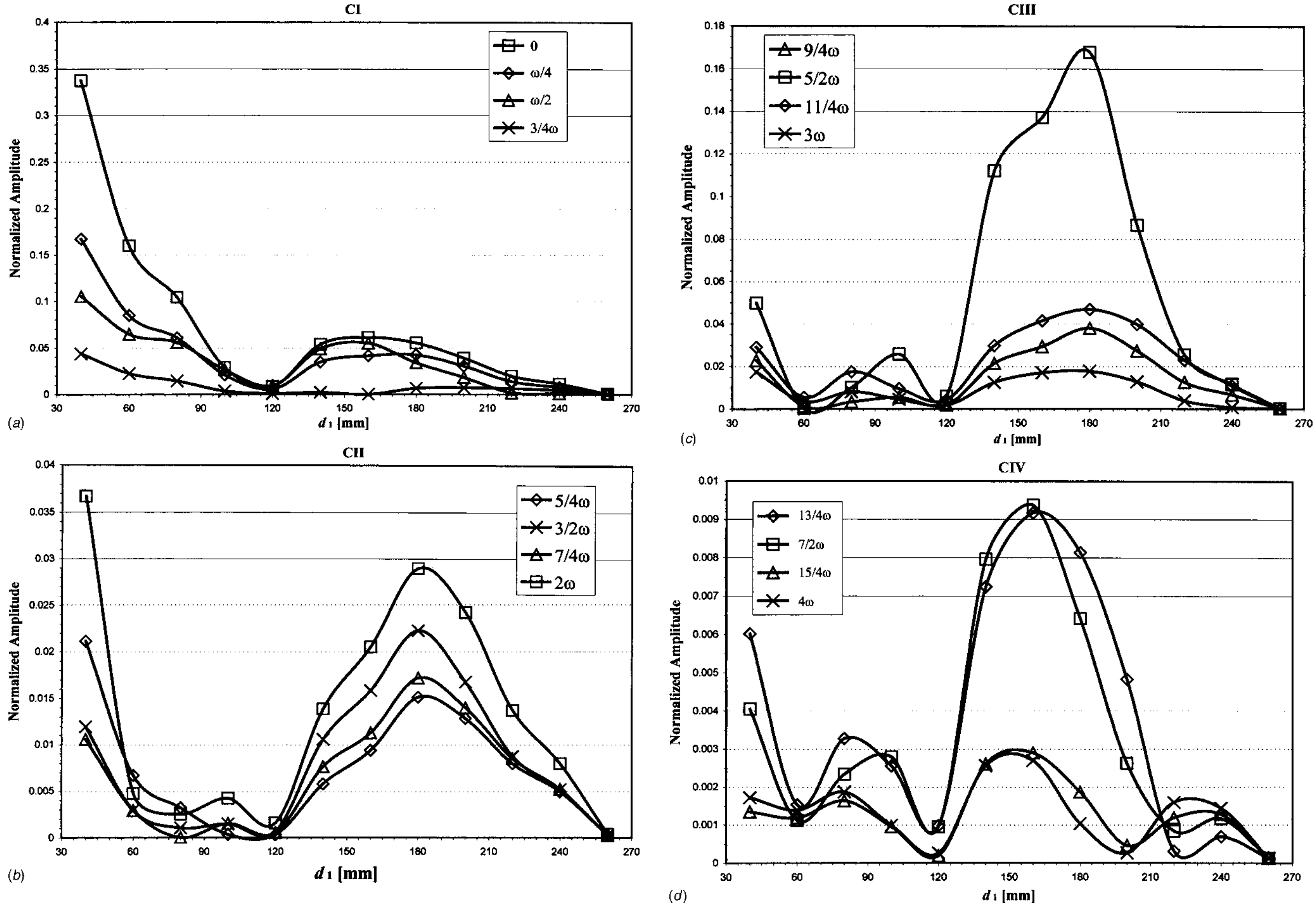

Fig. 4 One crack—Numerical simulations by varying the crack position (*a*) (CI). (*b*) (CII). (*c*) (CIII). (*d*) (CIV).

rameters of the simulations are summarized in Table 3. The numerical responses in terms of the normalized amplitude of Eq. (1) as a function of the crack position, are presented in Fig. 4. These diagrams clearly show that some particular crack positions, corresponding to a linear behavior, can be identified along the beam. These positions correspond to inflexion points in the beam elastic line, where the curvature is zero. In these positions the crack does not breath, so that it does not introduce a nonlinear behavior. For our cases, the inflexion point is between one-half and one-third of the beam length, starting from the clamp. Another inflexion point is clearly shown at the free-end of the beam: A crack placed in the extreme finite element does not change the linear behavior of the structure. This phenomenon can be used to detect the crack position. A real structure can be, in fact, monitored by varying the excitation (typically in terms of frequency). A linear behavior, corresponding to a particular value of the excitation frequency, implies a crack in the inflection point of the elastic line corresponding to that frequency. In the case considered in Fig. 4, the nonlinearity vanishes around the inflexion point corresponding to the second modal shape (consider that the first natural frequency of the undamaged structure is around one-half of that of excitation). In addition, Fig. 4 clearly shows that the nonlinearity increases if the distance between crack and clamp decreases, as previously observed combining simulations A and B. As a matter of fact, the sub-harmonic components can become predominant with respect to the super-harmonic ones.

2.4 Parametrical Simulations by Varying the Excitation Frequency (D). These numerical simulations consider one crack with a depth of $a_1=4.25$ mm. They are indicated by the letter D. Referring to Fig. 1, the coordinate of the crack is $d_1=90$ mm, the force amplitude is $F=2$N with a variable frequency f. The parameters of the simulations are summarized in Table 4. The numerical responses in terms of the normalized amplitude of Eq. (1) as a function of the excitation frequency are presented in Fig. 5. The most interesting result is that a particular harmonic component

Table 4 One crack—Numerical simulations by varying the amplitude of the excitation (D)

One crack—f variable $a_1=4.25$ mm; $d_1=90$ mm; $F=2$N	
f [Hz]	*Case*
2.0	D1
2.5	D2
3.3	D3
4.0	D4
5.0	D5
5.5	D6
6.5	D7
8.5	D8
11.0	D9
12.0	D10
13.0	D11
14.5	D12
15.0	D13
17.0	D14
18.0	D15
19.0	D16
19.5	D17
23.0	D18
25.0	D19
30.0	D20
34.0	D21
38.0	D22
49.8	D23

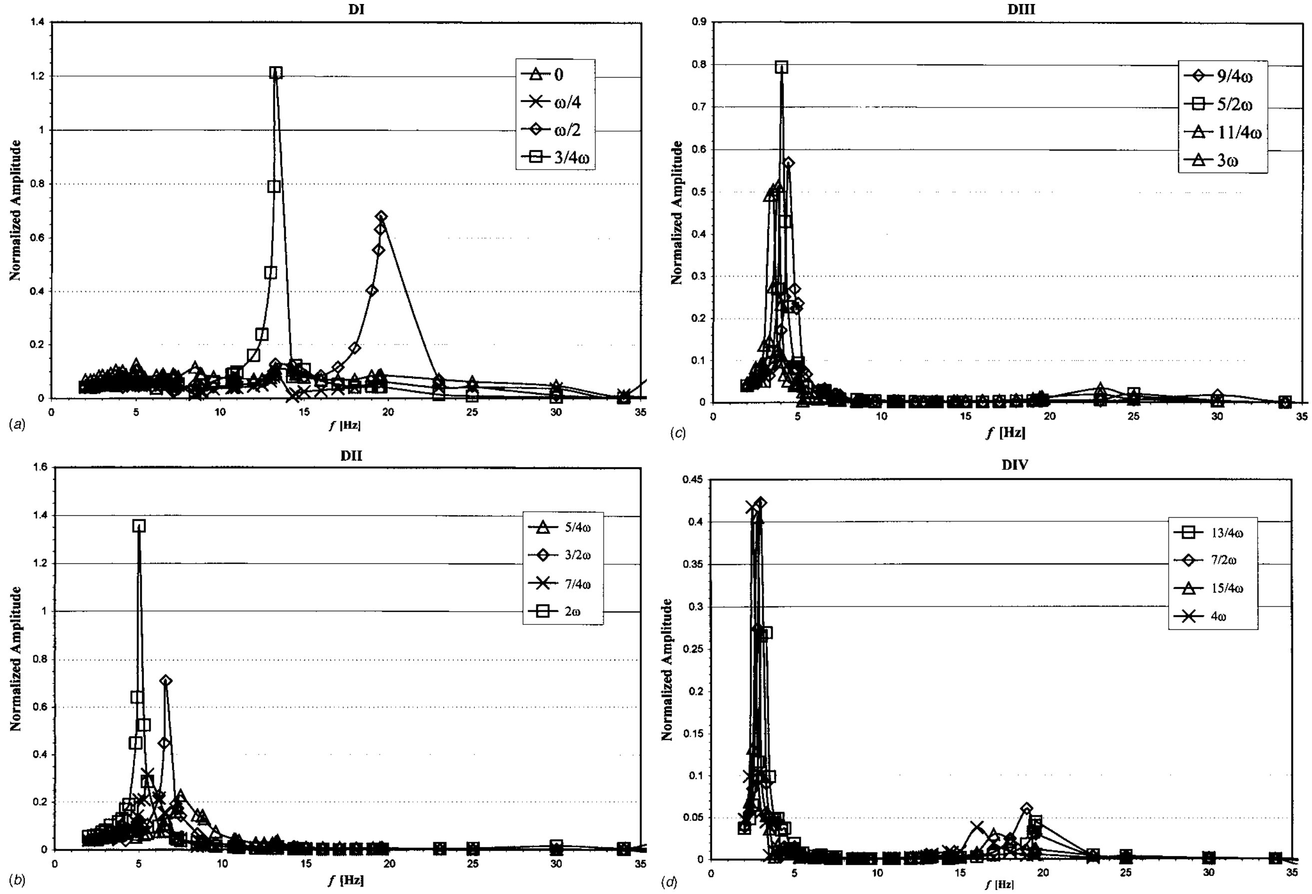

Fig. 5 One crack—Numerical simulations by varying the frequency of the excitation (*a*) (DI). (*b*) (DII). (*c*) (DIII). (*d*) (DIV).

becomes predominant in relation to its own resonance. This means that the component of frequency mf $(m=j/\Theta)$ becomes predominant when the frequency of the excitation satisfies:

$$f \approx \frac{f_0}{m} \tag{2}$$

f_0 being the first natural frequency of the damaged structure (in the present case it is around 9.5 Hz). Therefore, the $\frac{3}{4}\omega$ component, for example, goes into resonance around $f \approx 4/3 \times 9.5$ Hz ≈ 13 Hz, as well as the $\frac{1}{2}\omega$ component goes into resonance around $f \approx 2 \times 9.5$ Hz ≈ 19 Hz, according to the numerical results of Fig. 5(*a*). This phenomenon has been observed for all the frequency components.

2.5 Parametrical Simulations by Varying the Excitation Amplitude (E). These numerical simulations consider one crack with a depth of $a_1=4.25$ mm. They are indicated by the letter E. Referring to Fig. 1, the coordinate of the crack is $d_1=90$ mm, the excitation frequency is $f=12$ Hz and the force amplitude F is variable. The parameters of the simulations are summarized in Table 5. The numerical responses in terms of the normalized amplitude of Eq. (1) as a function of the frequency of the excitation are presented in Fig. 6. The results show that the stable solution is the trivial one of linearity with respect to the force amplitude. The existence of this linear solution appears rather obvious, as suggested by the motion equation reported in the companion paper (Part I).

2.6 Parametrical Simulations Considering Two Cracks and Varying the Depth of One of them (F). These numerical simulations consider two cracks, one of depth $a_1=4.25$ mm and the other of variable depth a_2. The simulations are indicated by the letter F. Referring to Fig. 1, the coordinates of the cracks are $d_1=90$ mm and $d_2=180$ mm, the force amplitude is $F=2$N with a frequency f of 19 Hz. The parameters of the simulations are summarized in Table 6. The numerical responses in terms of the normalized amplitude of Eq. (1) as a function of the crack depth are presented in Fig. 7. According to these diagrams, the nonlinearity seems to be less sensitive with respect to the crack depth a_2. This simply means that the predominant crack is the first one, since it is closer to the clamp. The trend changes only for very high depths a_2.

2.7 Parametrical Simulations Considering Two Cracks and Varying the Position of the Excitation (G). These numerical simulations consider two cracks, both of depth $a_1=a_2=4.25$ mm. The simulations are indicated by the letter G. Referring to Fig. 1, the coordinates of the cracks are $d_1=90$ mm and $d_2=180$ mm, the

Table 5 One crack—Numerical simulations by varying the amplitude of the excitation (E)

One crack—F variable $a_1=4.25$ mm; $d_1=90$ mm; $f=12$ Hz	
F [N]	*Case*
2.0	E1
1.0	E2
0.5	E3

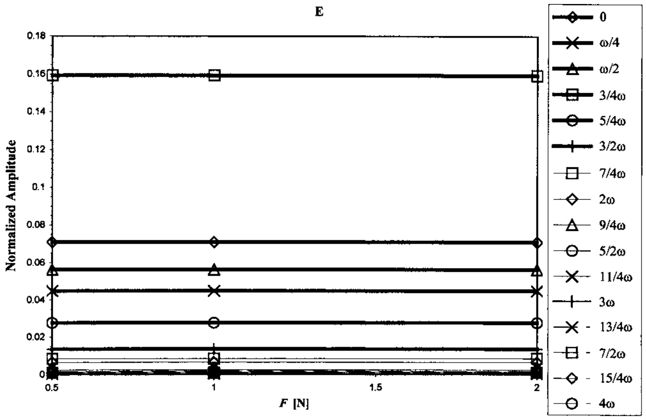

Fig. 6 One crack—Numerical simulations by varying the amplitude of the excitation (E)

Table 6 Two cracks—Numerical simulations by varying the depth of one crack (F)

Two cracks—a_2 variable a_1=4.25 mm; d_1=90 mm; d_2=180 mm; F=2N; f=19 Hz	
a_2 [mm]	*Case*
0.00	F1
1.00	F2
2.00	F3
2.20	F4
2.40	F5
2.60	F6
2.80	F7
3.00	F8
3.20	F9
3.40	F10
3.60	F11
3.80	F12
4.00	F13
4.20	F14
4.25	F15
4.40	F16

force amplitude is F=2N, with a frequency f of 19 Hz and the position of the excitation d_F is variable. The parameters of the simulations are summarized in Table 7. The numerical responses in terms of the normalized amplitude of Eq. (1) as a function of the position of the excitation are presented in Fig. 8. According to these diagrams, the nonlinearity presents a clear transition between the two cracks, larger near the first than near the second one. The largest nonlinearities arise for values of the force position between the first crack and the clamp. A very interesting result is that the stronger nonlinearity appears for excitations near the position of the predominant crack. It is important to note that the trend does not change substantially when the force position is closer to the clamp (consider that these are contributions normalized with respect to the linear one, see Eq. (1), so that they do not vanish near the clamp).

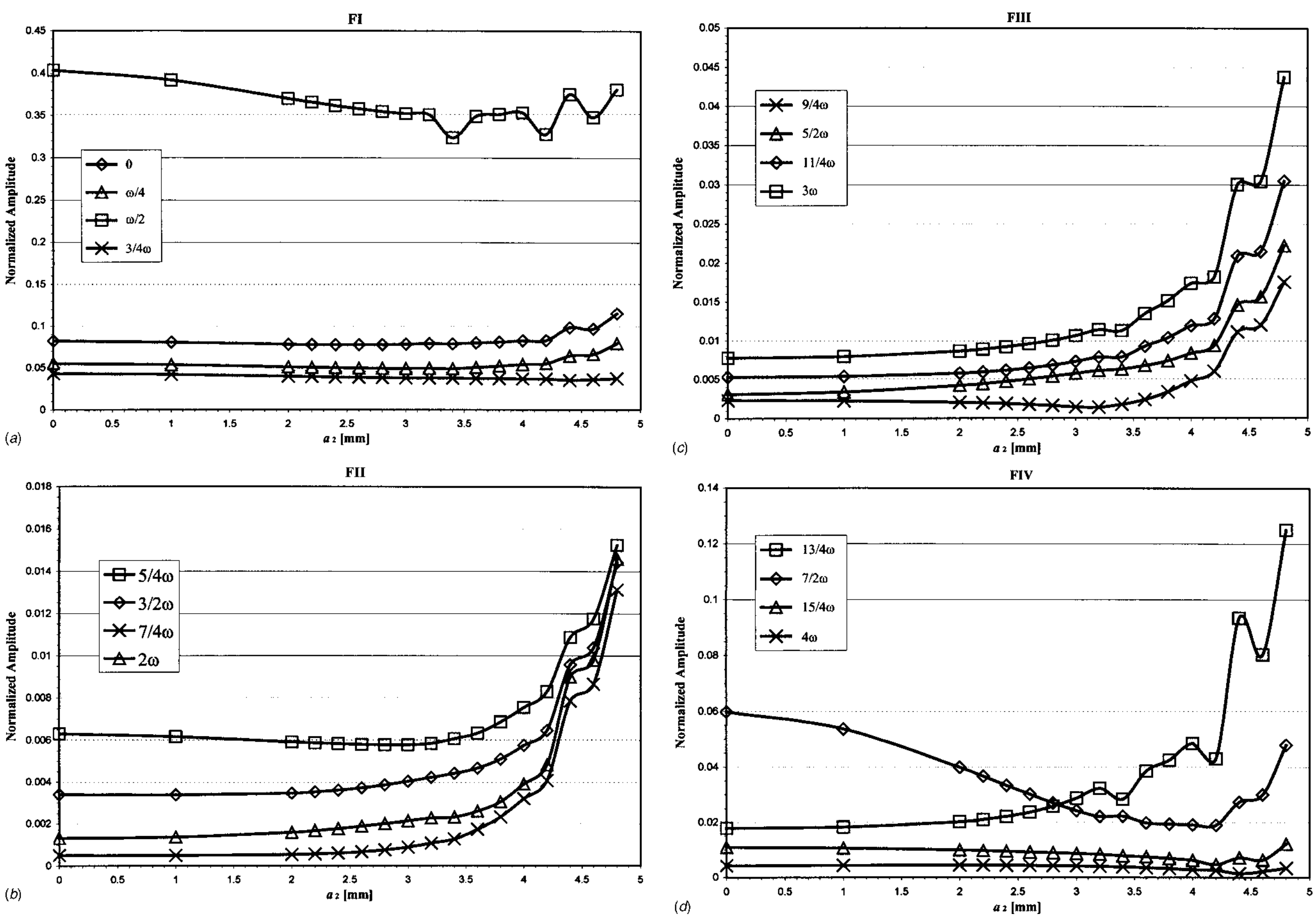

Fig. 7 Two cracks—Numerical simulations by varying the depth of one crack (*a*) (FI). (*b*) (FII). (*c*) (FIII). (*d*) (FIV).

Table 7 Two cracks—Numerical simulations by varying the position of the excitation (G)

Two cracks—d_F variable $a_1 = a_2 = 4.25$ mm; $d_1 = 90$ mm; $d_2 = 180$ mm; $F = 2$N; $f = 19$ Hz	
d_F [mm]	*Case*
270.0	G1
189.0	G2
135.0	G3
94.5	G4
13.5	G5

3 General Discussion

The theoretical and numerical approach presented in the companion paper (Part I) appears very useful in the study of highly nonlinear forced vibrations for damaged structures. It permits us to take into account the interaction of several breathing cracks. In the case of high nonlinearity, the super-harmonic frequency components become insufficient to catch the real behavior of the structure. As a consequence (offset and) sub-harmonic components must be taken into account. One example is given by the period doubling phenomenon, recently experimentally observed and discussed in both Parts I and II. The extensive parametrical simulations, presented in Part II, have been performed by varying all the main parameters influencing the dynamic behavior of the structure: The number of cracks, their depth and position, as well as the amplitude, frequency, and position of the excitation.

The results can be summarized as follows:

(1) For a weak nonlinearity, we have to take into account in the structural response not only the super-harmonic frequency components but also an offset (zero-frequency);
(2) for a stronger nonlinearity, we have to take into account not only the super-harmonic frequency and offset components but also the sub-harmonic ones (complexity and transition to deterministic chaos);
(3) if a crack implies, as a particular case, a linear behavior of the structure, we can conclude that the crack position is close to an inflection point of the elastic line corresponding to the excitation frequency. This result could be useful to improve the techniques of vibration-based inspection;
(4) the nonlinearity increases if the position of the excitation becomes closer to the crack (or, obviously, if the crack position becomes closer to the clamp, or if the crack depth becomes larger);
(5) the component of frequency mf becomes predominant in correspondence of its own resonance, when the frequency of the excitation satisfies $f \approx f_0/m$;
(6) the behavior becomes linear with respect to the amplitude of the excitation.

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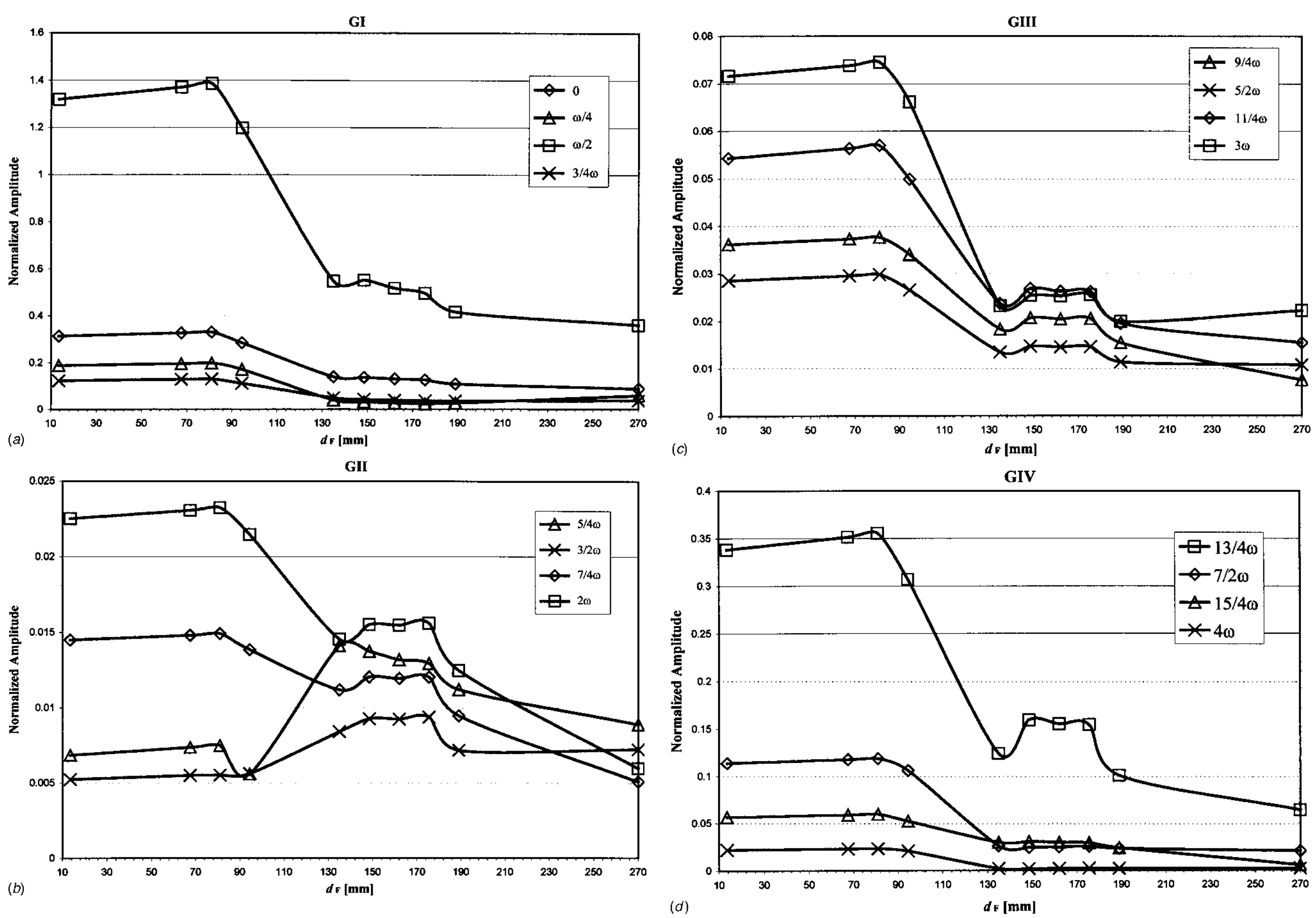

Fig. 8 Two cracks—Numerical simulations by varying the position of the excitation (*a*) (GI). (*b*) (GII). (*c*) (GIII). (*d*) (GIV).

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