Towards Chaos in Vibrating Damaged Structures—Part I: Theory and Period Doubling Cascade

Alberto Carpinteri

Nicola Pugno

Department of Structural Engineering and Geotechnics,
Politecnico di Torino,
Corso Duca degli Abruzzi 24,
10129 Torino, Italy

The aim of the present paper is to evaluate the complex oscillatory behavior, i.e., the transition toward deterministic chaos, in damaged nonlinear structures under excitation. In the present paper (Part I), we show the developed theoretical approach and how it allows us to capture not only the super-harmonic and offset components (predominant for moderate nonlinear systems) but also the subharmonics of the structural dynamic response, describing complex and highly nonlinear phenomena, like the experimentally observed period doubling. Moreover, a period doubling cascade with a route to chaos seems to emerge from our considerations. [DOI: 10.1115/1.1934582]

1 Introduction

Vibration-based inspection of structural behavior offers an effective tool of non-destructive testing. The analysis of the dynamic response of a structure to excitation forces and the monitoring of alterations which may occur during its lifetime can be employed as a global integrity-assessment technique to detect, for example, play in joints or the presence of a crack. Indeed it is well known that, in the case of simple structures, crack position and depth can be determined from changes in natural frequencies, modes of vibration or the amplitude of the forced response.

The damage assessment problem in cracked structures has been extensively studied in the last decade, highlighting that the vibration based inspection is a valid method to detect, localize, and quantify cracks especially in beam structures. Dealing with the presence of a crack in the structure, previous studies have demonstrated that a transverse crack can change its state (from open to closed and vice versa) when the structure, subjected to an external load, vibrates. As a consequence, a nonlinear dynamic behavior is introduced.

In the past many studies have illustrated that a crack in a structure such as a beam, may exhibit nonlinear behavior if it is open during part of the response and closed in the remaining intervals. This phenomenon has been detected during experimental testing performed by Gudmundson [1] in which the influence of a transverse breathing crack upon the natural frequencies of a cantilever beam was investigated. The main result obtained was that the experimentally observed decrease in the natural frequencies of the beam due to the presence of the crack was not sufficient to be described by a model of crack which is always open. Therefore, it must be concluded that the crack alternately opened and closed giving rise to natural frequencies falling between those corresponding to the always-open and always-closed (e.g., integral) cases. In fact, if an always-open crack is assumed in the analysis of a beam with a so-called breathing crack, which both opens and closes during the time interval considered, the reduced decrease in the experimental natural frequencies will lead to an underestimate of the crack depth if determined via a test-model correlation approach.

Friswell and Penny [2] have simulated the nonlinear behavior of a beam with a breathing crack vibrating in its first mode of vibration through a simple one-degree-of-freedom model with bilinear stiffness. The analysis of the fast Fourier transform and the response to harmonic loading, obtained by numerical integration, demonstrates the occurrence of harmonics in the response spectra which are integer multiples of the exciting frequency.

Krawczuk and Ostachowicz [3,4] presented an analysis of the forced vibrations of a cantilevered beam with a breathing crack, in which the equations of motion are solved using the harmonic balance technique. The periodically time-variant beam stiffness is simulated by a square wave function with a fundamental frequency equal to the forcing term frequency. According to [2] this research has shown that, when a breathing crack is present in a beam, higher harmonic components in the frequency spectrum of the response are generated if excited by a sinusoidal forcing function, indicating that the structure behaves nonlinearly.

Crespo, Ruotolo, and Surace [5] have solved and discretized the nonlinear equation of motion of a beam with breathing crack using the finite element method. According to the cited papers, the main assumption has been that the crack can be either fully open or fully closed during the vibration.

In [6] the vibrational response of a cantilevered beam with closing crack to harmonic forcing has been analyzed and its dynamic behavior characterized by using the so-called higher order frequency response functions.

Carpinteri and Carpinteri [7] highlight how in reality the crack opening and closing are continuous phenomena, i.e., the crack can be even partially open (or closed) as a function of the cracked element curvature.

The aim of this paper is to develop a coupled theoretical and numerical approach to evaluate the complex oscillatory behavior in damaged nonlinear structures under excitation. In particular, we have focused our attention on a cantilever beam with several breathing transverse cracks and subjected to harmonic excitation perpendicular to its axis. The method, that is an extension of the super-harmonic analysis carried out by Pugno, Surace, and Ruotolo [8] to subharmonic and zero frequency components, has allowed us to capture the complex behavior of the nonlinear structure, e.g., the occurrence of *period doubling*, as experimentally observed by Brandon and Sudraud [9] in cracked beams. The first results of this approach have been presented by Carpinteri and

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Pugno in [10]. The method described assumes, as previously done in [10] and according to [7], that the cracks open and close continuously instead of instantaneously, as suggested by the experiments. As a consequence, the cracks are not considered to be either fully open or fully closed, but the intermediate configurations with partial opening have also been taken into account.

The period of the response is not assumed a priori equal to the period of the harmonic excitation, as classically supposed (absence of subharmonic components). This has allowed us to capture the complex behavior of the highly nonlinear structure, e.g., the occurrence of period doubling.

A pioneer work on period doubling was written in 1978, when Mitchell Feigenbaum [11] developed a theory to treat its route from the ordered to chaotic states. Even if oscillators showing the period doubling can be of different nature, as in mechanical, electrical, or chemical systems, these systems all share the characteristic of recursiveness. He provided a relationship in which the details of the recursiveness become irrelevant, through a kind of universal value, measuring the ratio of the distance between successive period doublings, the so called *Feigenbaum's delta* [12]. His understanding of the phenomenon was later experimentally confirmed [13], so that today we refer to the so-called *Feigenbaum's period doubling cascade*. However, even if the period doubling has a long history, only recently it has been experimentally observed in dynamics of cracked structures [9]. The aim of our study is the understanding of such phenomenon that, according to our model, seems to be ruled by the breathing of the cracks during the oscillation of the structure.

In addition to the super-harmonics, the analysis has systematically emphasized a presence of an offset (zero frequency component) in the structural response also for weak nonlinearities. Furthermore, subharmonic components appear in the response of the structure for stronger nonlinearities, leading, in particular conditions, to the period doubling.

The differential nonlinear equations governing the oscillations of the continuum structure, discretized by the finite element method, have been analyzed by means of the Fourier transforms or Fourier trigonometric series coupled with the harmonic balance approach. This allows us to obtain a nonlinear system of algebraic equations, easy to be solved numerically. In the numerical examples, the phenomenon of the period doubling is discussed, with an eye to the phase space trajectories.

2 Theoretical Continuum Approach

Let us consider a multicracked cantilever beam, clamped at one end and subjected to a dynamic distributed force p (with rotating frequency ω). Modeling the breathing cracks as concentrated nonlinear compliances (or stiffnesses) (in [14] linear stiffnesses are assumed), the equation of the motion of each integer beam segment, is the classical equation of the beam dynamics. Furthermore, the boundary conditions between two adjacent segments are represented by the continuity of the transversal displacement and of its second and third spatial derivatives (proportional to the bending moment and to the shearing force respectively), as well as by the compatibility with the crack. This implies that the difference in the rotations between the two adjacent sections must be equal to the rotation of the connecting concentrated nonlinear stiffness. The problem formally can be written as

$$\rho A \frac{\partial^2 q(z,t)}{\partial t^2} + EI \frac{\partial^4 q(z,t)}{\partial z^4} = p(z,t) \text{ for } z_i < z < z_{i+1};$$

whereas for $z = z_i$:

$$q(z_i^-) = q(z_i^+),\ q''(z_i^-) = q''(z_i^+),\ q'''(z_i^-) = q'''(z_i^+),\ q'(z_i^+) - q'(z_i^+) = \frac{EIq''(z_i^\pm)}{k_i[q'(z_i^\pm)]} \tag{1}$$

where ρ is the density, A the cross-section area, q the transversal displacement, E the Young's modulus, and I the moment of inertia of the beam. k_i is the nonlinear concentrated rotational stiffness (a function of the rotations $q'(z_i^\pm)$) of the crack placed at z_i (the symbol prime denotes the derivation with respect to z). We will discuss the form of the nonlinear stiffness in the next section, according to the experimental evidence.

Equation (1) can be formally solved by applying the Fourier trigonometric series searching a solution in the form

$$q(z,t) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k(z) e^{ik\omega/\Theta t}, \tag{2}$$

where $c_k(z)$ are unknown functions and $\tilde{P} = \Theta 2\pi/\omega = \Theta P$ is the period of the response, assumed a priori to be different from the period P of the excitation (describing a so-called complex behavior, thus we call Θ complexity index). On the other hand, if the period of the response tends to infinite, i.e., $\Theta \to \infty$ (nonperiodic response, i.e., chaotic deterministic behavior), it is well-known that Eq. (2) becomes formally a Fourier transform, i.e.

$$q(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega t} Q(z,\omega) \mathrm{d}\omega, \tag{3}$$

where $Q(z,\omega)$ is the new unknown function. Thus, an approximation of the continuum spectrum in the response $q(z,t)$ of Eq. (3) is represented by the discrete spectrum of Eq. (2) if a sufficiently large complexity index Θ is considered. In addition, instead of the continuum approach of Eq. (1), a discretization of the system could also be considered. These discretizations allow an easier solution of the problem, as we will point out in the next section.

3 Theoretical Discrete Approach

By discretizing the structure with the finite element method [8], Eq. (1) can be rewritten as

$$[\mathrm{M}]\{\ddot{q}\} + [\mathrm{D}]\{\dot{q}\} + [\mathrm{K}]\{q\} + \sum_m [\Delta \mathrm{K}^{(m)}] f^{(m)}(\{q\})\{q\} = \{F\}, \tag{4}$$

where [M] is the mass matrix, [D] the damping matrix, [K] $+\Sigma_m[\Delta \mathrm{K}^{(m)}]$ the stiffness matrix of the undamaged beam, and $[\Delta \mathrm{K}^{(m)}]$ is half of the variation in stiffness introduced when the mth crack is fully open (see the Appendix and [8]). $\{F\}$ is the vector of the applied forces (with angular frequency ω) and $\{q\}$ is the vector containing the generalized displacements of the nodes (translations and rotations). According to this notation, $f^{(m)}(\{q\})$ is between -1 and $+1$ and models the transition between the conditions of mth crack fully open and fully closed. Assuming that this transition is instantaneous and hence takes place discontinuously, $f^{(m)}(\{q\})$ is a step function and has the sign of the curvature of the corresponding cracked element. With this simple model of crack opening and closing, $f^{(m)}(\{q\})$ can thus only be equal to -1 or $+1$. On the other hand, in the present investigation as in the previous [8], $f^{(m)}(\{q\})$ is assumed to be a linear function of the curvature of the corresponding cracked element. In other words, the cracks are not considered fully open or fully closed, as the intermediate configurations with partial opening are also taken into account. Thus, the stiffness varies continuously between the two extremes of undamaged or totally damaged beam (fully open cracks), rather than stepwise. The solution for the elements of the $\{q\}$ vector $\in L^2$ (i.e., $|q_i|^2$ can be integrated according to Lebesgue) can be found by means of Eq. (3), or by the approximation of Eq. (2), that for our discrete system can be rewritten as

$$q_i = \sum_{j=0}^{N} \left(A_{ij} \sin j\frac{\omega}{\Theta}t + B_{ij} \cos j\frac{\omega}{\Theta}t \right), \tag{5}$$

in which the complexity integer Θ must be a positive integer, to take into account not only the super-harmonics (and offset) but also the subharmonic components of the dynamic response and theoretically $N=\infty$. As previously discussed, this means that the response could have a period $\tilde{P}=\Theta P$ that is not a priori coincident with the period P of the excitation. A value for Θ tending to infinite (Fourier trigonometric series become Fourier transforms) describes a transition toward a chaotic (nonperiodic) response.

It is interesting to note that, even if the trigonometric series (2) converges, it could not be a trigonometric Fourier series. In fact, the Fischer-Riesz theorem affirms that it is a fourier series if and only if $\Sigma_{j=0}^{\infty}(|A_{ij}|^2+|B_{ij}|^2)$ converges. In this case, the Parseval equation

$$\sum_{j=0}^{\infty}(|A_{ij}|^2+|B_{ij}|^2)=\frac{2}{\tilde{P}}\int_{-\tilde{P}/2}^{+\tilde{P}/2}|q_i(t)|^2\mathrm{d}t, \tag{6}$$

obviously implies

$$\lim_{j\to\infty}A_{ij}=\lim_{j\to\infty}B_{ij}=0. \tag{7}$$

The last relationships allow us to consider a finite number N in Eq. (5), large enough to provide a good approximation. The function $f^{(m)}(\{q\})$ is considered linear versus the curvature of the corresponding cracked element, i.e.,

$$f^{(m)}(\{q\})=\frac{q_{m_k}-q_{m_h}}{|q_{m_k}-q_{m_h}|_{\max}}=\Lambda_m(q_{m_k}-q_{m_h}), \tag{8}$$

where the numerator represents the difference in the rotations at the ends of the corresponding cracked element and the denominator is the maximum absolute value that can be reached by this difference: consequently, the generic component of function $\{g^{(m)}\}=f^{(m)}(\{q\})\{q\}$ (that appears in Eq. (4)) can be expressed as

$$g_i^{(m)}=\Lambda_m(q_{m_k}-q_{m_h})q_i. \tag{9}$$

The same concepts argued for the q_i components can be now applied to the $g_i^{(m)}$, ensuring that they can be developed in a trigonometric Fourier series and can thus be put in the approximate form

$$g_i^{(m)}=\sum_{j=0}^{N}\left(C_{ij}^{(m)}\sin j\frac{\omega}{\Theta}t+D_{ij}^{(m)}\cos j\frac{\omega}{\Theta}t\right), \tag{10}$$

with

$$C_{ij}^{(m)}=\frac{2}{\tilde{P}}\int_0^{\tilde{P}}g_i^{(m)}(t)\sin\left(j\frac{\omega}{\Theta}t\right)\mathrm{d}t, \tag{11}$$

$$D_{ij}^{(m)}=\frac{2}{\tilde{P}}\int_0^{\tilde{P}}g_i^{(m)}(t)\cos\left(j\frac{\omega}{\Theta}t\right)\mathrm{d}t. \tag{12}$$

Inserting relation (9), in its explicit form according to Eq. (5) for the degrees of freedom i, m_h, and m_k, into Eqs. (11) and (12) and developing the integrals, gives the following expressions:

$$C_{ij}^{(m)}=\frac{\Lambda_m}{2}\sum_{j_1,j_2:j_1+j_2=j}\{(A_{m_kj_1}-A_{m_hj_1})B_{ij_2}+(B_{m_kj_1}-B_{m_hj_1})A_{ij_2}\} + \frac{\Lambda_m}{2}\sum_{j_1,j_2:j_1-j_2=\pm j}\pm\{(A_{m_kj_1}-A_{m_hj_1})B_{ij_2}-(B_{m_kj_1}-B_{m_hj_1})A_{ij_2}\}, \tag{13}$$

$$D_{ij}^{(m)}=\frac{\Lambda_m}{2}\sum_{j_1,j_2:j_1+j_2=j}\{-(A_{m_kj_1}-A_{m_hj_1})A_{ij_2}+(B_{m_kj_1}-B_{m_hj_1})B_{ij_2}\} + \frac{\Lambda_m}{2}\sum_{j_1,j_2:j_1-j_2=\pm j}\pm\{(A_{m_kj_1}-A_{m_hj_1})A_{ij_2}-(B_{m_kj_1}-B_{m_hj_1})B_{ij_2}\}. \tag{14}$$

As the nonlinearity of the components of $\{g^{(m)}\}=f^{(m)}(\{q\})\{q\}$ were expressed in a form analogous to that of the components of $\{q\}$, as indicated by Eq. (10), it is possible at this stage to apply the harmonic balance method (Eqs. (5) and (10)—with Eqs. (13) and (14)—must be introduced in Eq. (4) and the terms of the harmonics of the same order must be "balanced"), which leads to N different systems of nonlinear algebraic equations. We find the following solution:

$$\begin{bmatrix}[\mathrm{K}]-\frac{j^2\omega^2}{\Theta^2}[\mathrm{M}] & -\frac{j\omega}{\Theta}[\mathrm{D}]\\ \frac{j\omega}{\Theta}[\mathrm{D}] & [\mathrm{K}]-\frac{j^2\omega^2}{\Theta^2}[\mathrm{M}]\end{bmatrix}\begin{Bmatrix}\{A_j\}\\ \{B_j\}\end{Bmatrix} =\{F_j\}-\sum_m\begin{bmatrix}[\Delta\mathrm{K}^{(m)}] & [0]\\ [0] & [\Delta\mathrm{K}^{(m)}]\end{bmatrix}\begin{Bmatrix}\{C_j^{(m)}\}\\ \{D_j^{(m)}\}\end{Bmatrix}, \tag{15}$$

where $j=0,1,\ldots,N$ and for each vector we have $\{V_j\}^T=\{V_{1j},V_{2j},\ldots,\}^T$. In addition

$$F_{ij}=F\delta_{j\Theta}\delta_{ip}, \tag{16}$$

p being the node position corresponding to the point where the sinusoidal force is applied.

Each system can be solved numerically using an iterative procedure interrupted by an appropriate convergence test when the relative j error for the $\{A_j\}$ and $\{B_j\}$ vectors becomes lower than a specified value; it is a function of the kth iteration and has been defined as

$$e_{jk}=\left\|\begin{Bmatrix}\{A_j\}\\ \{B_j\}\end{Bmatrix}_k-\begin{Bmatrix}\{A_j\}\\ \{B_j\}\end{Bmatrix}_{k-1}\right\|\Bigg/\left\|\begin{Bmatrix}\{A_j\}\\ \{B_j\}\end{Bmatrix}_{k-1}\right\|. \tag{17}$$

The procedure consists in determining the unknowns A_{ij} and B_{ij}. It is very interesting to note that, assuming the coefficients $C_{ij}^{(m)}$, $D_{ij}^{(m)}$ to be zero at the first step, implies to force also the subharmonic components to be zero (see Eq. (15)). So, differently from the super-harmonic analysis [8], we have to start with nonzero values for the coefficients $C_{ij}^{(m)}$, $D_{ij}^{(m)}$. To obtain good initial values for these coefficients, we have considered as a zero step a super-harmonic analysis ($\Theta=1$); in this case, we can determine the unknowns A_{ij} and B_{ij} starting with zero coefficients $C_{ij}^{(m)}$, $D_{ij}^{(m)}$ and, by Eqs. (13) and (14), we have their initial values for the subharmonic analysis. The solution thus obtained is used to determine the known vector of the right hand-side of Eq. (15). The procedure is repeated until the desired precision is achieved and coefficients A_{ij} and B_{ij} are found, while Eq. (5) is used to determine the components of the approximate vector, which satisfies the nonlinear Eq. (4), giving an approximated solution of the continuum system described in Eq. (1).

4 Period Doubling Cascade

We can consider two different numerical examples: a weakly nonlinear structure and a strongly nonlinear one. Only in the latter case, the so-called period doubling phenomenon, experimentally observed by Brandon and Sudraud [9], clearly appears. The beam here considered is the same as that described in the mentioned experimental analysis. It is 270 mm long and has a transversal rectangular cross section of base and height, respectively, of 13 and 5 mm. The material is (UHMW)-ethylene, with a Young's modulus of 8.61×10^8 N/m^2 and a density of 935 kg/m^3. We have assumed a modal damping of 0.002. It is discretized with 20

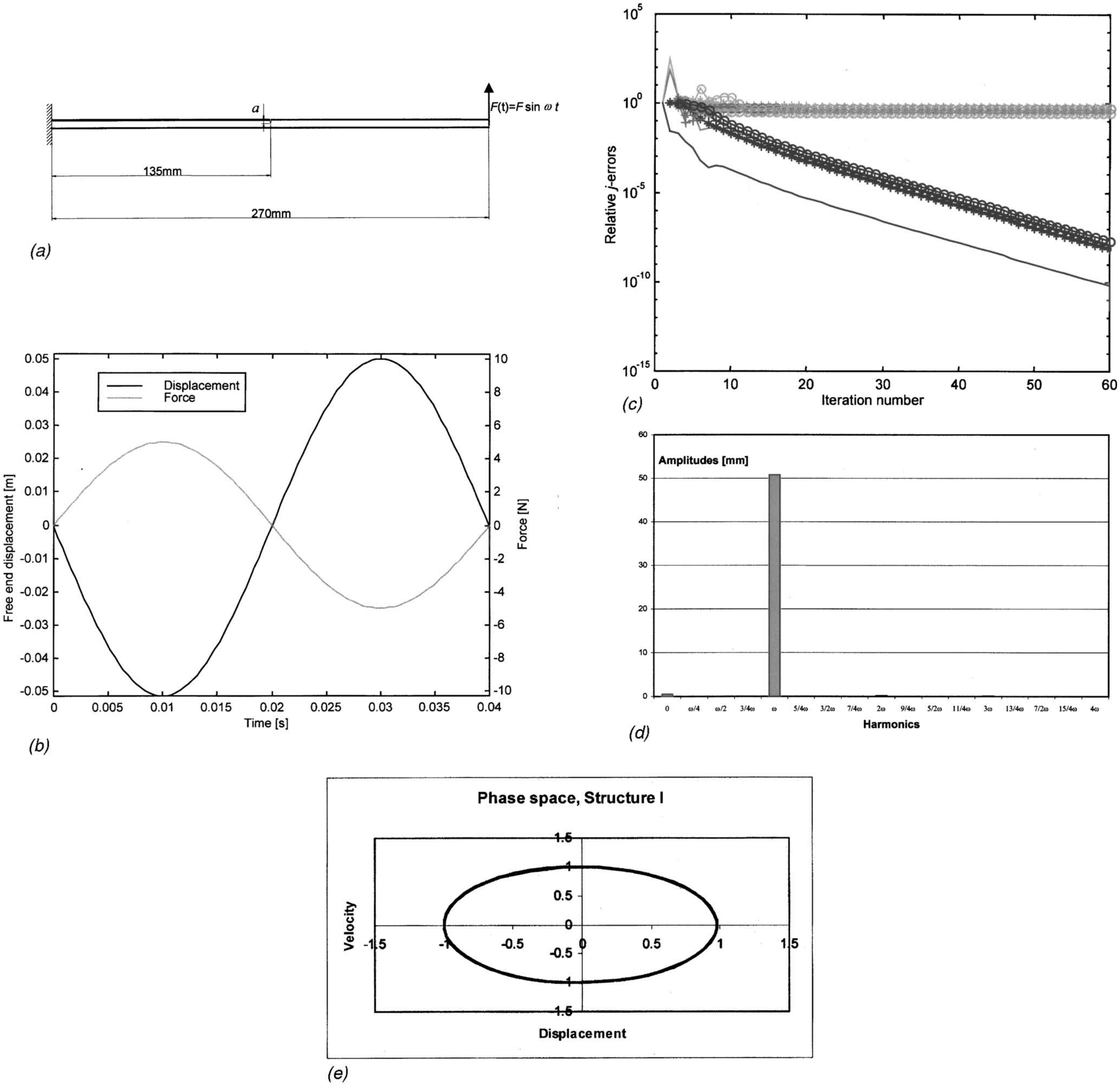

Fig. 1 (*a*) **Structure I—Damaged structure and characteristics of the excitation** ($a=2.4$ **mm,** $F=5$**N,** $f=\omega/2\pi=25$ **Hz).** (*b*) **Structure I—Time history of the free end displacement and of the applied force.** (*c*) **Structure I—Relative error as a function of the iteration number, for each** j**th harmonic** ($j=0,1,\ldots,16$)**.** (*d*) **Structure I—Zero- (offset), sub- and super-harmonic components for the free end displacement (i.e.,** $\sqrt{A_{20j}^2+B_{20j}^2}$ **for** $j=0,1,\ldots,16$**).** (*e*) **Structure I—Dimensionless phase diagram of the response (free-end displacement).**

finite elements. We have found that a complexity index $\Theta=4$ and $N=16$ give a good approximation. For larger values of Θ and N, substantially coincident solutions are obtained. The first natural frequency of the undamaged structure is $f_u=10.6$ Hz.

For each of the two considered structures (Figs. 1(*a*) and 2(*a*)) it is shown the time history of the applied force and of the free-end displacement (Figs. 1(*b*) and 2(*b*)), the relative errors as functions of the iteration number (Figs. 1(*c*) and 2(*c*)) and the zero-, sub-, and super-harmonic components for the free-end displacement (Figs. 1(*d*) and 2(*d*)). In Tables 1 and 2, the frequency components are considered separately as sin and cos components.

In a hypothetical linear structure, the structural response is linear by definition with obviously only one harmonic component at the same frequency of the excitation.

In the weakly nonlinear structure of Fig. 1(*a*), the response converges and it appears only weakly nonlinear, as depicted in Fig. 1(*b*). The relative errors, shown in Fig. 1(*c*), tend to zero or by definition are equal to 1 if related to the harmonic components identically equal to zero. The harmonic components in the structural response are the zero-one (presence of a negative offset in the displacement of the free-end, downward in Fig. 1(*a*)) and the super-harmonic ones (Fig. 1(*d*) and Table 1). No subharmonic components can be observed. The corresponding phase diagram of the response is shown in Fig. 1(*e*). Due to the weak nonlinearity the trajectory in the phase diagram is close to an ellipse. The diagram is nonsymmetric as the spatial positions of the cracks (placed in the upper part of the beam). The trajectory is an unique closed curve since here the period of the response is equal to the

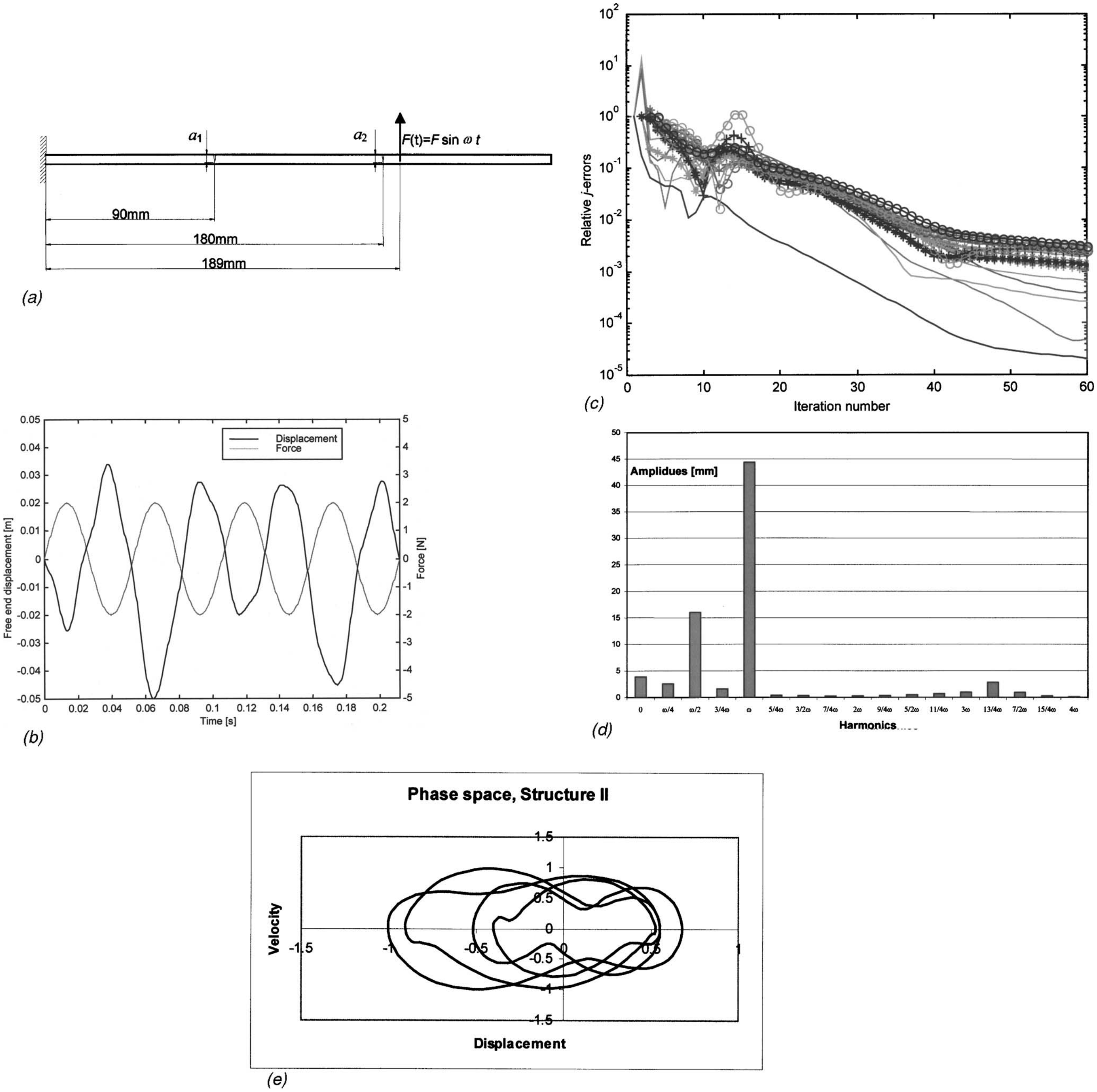

Fig. 2 (*a*) Structure II—Damaged structure and characteristics of the excitation (a_1=4.25 mm, a_2=4.25 mm, F=2N, $f=\omega/2\pi$ =18.9 Hz). (*b*) Structure II—Time history of the free end displacement and of the applied force. (*c*) Structure II—Relative error as a function of the iteration number, for each *j*th harmonic (j=0,1,…,16). (*d*) Structure II—Zero- (offset), sub- and super-harmonic components for the free end displacement (i.e., $\sqrt{A_{20j}^2+B_{20j}^2}$ for j=0,1,…,16). (*e*) Structure II—Dimensionless phase diagram of the response (free-end displacement).

period of the excitation. An increasing of the nonlinearity will distort the trajectory as we will show in the next example.

In the strongly nonlinear structure of Fig. 2(*a*) the response converges (Fig. 2(*c*)) and the nonlinearity increases, as depicted in Fig. 2(*b*). The harmonic components in the structural response are the zero one, the super harmonic as well as the subharmonic ones (Fig. 2(*d*) and Table 2). It should be emphasized that a strong presence of the component causes the period doubling of the response, i.e., the $\omega/2$ component. The free-end vibrates practically with a period doubled with respect to the excitation. A nonnegligible component at $\omega/4$ is observed too, representing a route to chaos through a period doubling cascade.

The corresponding phase diagram of the response is shown in Fig. 2(*e*). The trajectory is a unique closed curve since the response is still periodic; it is composed by multiple cycles since here the period of the response is not equal to the period of the excitation. The distortions in the trajectory are consequences of the presence of the super- or subharmonics, as well as the multiple cycles emphasize the presence of the subharmonics (four cycles are due to the component $\omega/4$), i.e., the presence of a complexity with associated route to chaos. Also in this case, the diagram is nonsymmetric as the spatial positions of the cracks.

An extensive parametrical investigation can be found in the companion paper.

Table 1 Structure I—Zero- (offset), sub- and super-harmonic sin- and cos- components [mm], for the free end displacement (i.e., A_{20j}, B_{20j}, for $j=0, 1, \ldots, 16$)

Harmonic	Sin	Cos	Amplitude
0	0	−0.5472	0.5472
$\omega/4$	0	0	0
$\omega/2$	0	0	0
$3/4\omega$	0	0	0
ω	−50.7971	−0.1414	50.7973
$5/4\omega$	0	0	0
$3/2\omega$	0	0	0
$7/4\omega$	0	0	0
2ω	0.0039	0.1688	0.1689
$9/4\omega$	0	0	0
$5/2\omega$	0	0	0
$11/4\omega$	0	0	0
3ω	0.0745	−0.0008	0.0745
$13/4\omega$	0	0	0
$7/2\omega$	0	0	0
$15/4\omega$	0	0	0
4ω	0.0001	0.0035	0.0035

5 Conclusions

The proposed approach extends the theory proposed by Pugno et al. [8] to (offset and) subharmonic components. We have demonstrated that our approach corresponds to an approximated solution of the continuum spectrum of the response of the continuum system. The method has allowed us to catch complex phenomena, i.e., transition toward deterministic chaos, like the occurrence of a period doubling, as shown in the numerical examples and experimentally observed in the context of cracked beam by Brandon and Sudraud [9]. In this analysis, of crucial importance appears the complexity index Θ. For higher values of Θ we have to increase also N (e.g., $N \approx \Theta^2$), so that the complexity of the numerical simulations considerably increases. On the other hand, larger values of Θ allow us to catch higher structural complexity, as emphasized by multiple cycles of the trajectory in the phase space diagrams.

From the reported numerical examples (for an extensive numerical parametrical investigation see the companion paper), we can affirm that if the nonlinearity is zero, the structural response (i.e., Eq. (5)) can be obviously caught with $N=\Theta=1$. If a weak nonlinearity is considered, only offset and super-harmonic components can be observed in the structural response. As a consequence, for this case, it can be easily caught using classical Fourier series ($\Theta=1$) with $N>1$ (and large enough, in the sense that Response(N)$\cong$Response($N'>N$)). If the nonlinearity becomes stronger, offset, and super-harmonic components, as well as subharmonic ones, can be observed in the structural response. As a consequence, in this case, it can be caught using a complex index Θ larger than 1 ($\Theta>1$ and $N \gg 1$ large enough, in the sense that response(Θ,N)$\cong$response($\Theta'>\Theta$,$N'>N$)). Theoretically, values of Θ tending to infinity (Fourier series become Fourier transforms, with theoretically N tending to infinity too) allow us to catch a route to chaos through a period doubling cascade, that here would imply a nonperiodic dynamic response. These considerations are summarized in Table 3.

Acknowledgment

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Table 2 Structure II—Zero- (offset), sub- and super-harmonic sin- and cos-components [mm], for the free end displacement (i.e., A_{20j}, B_{20j}, for $j=0, 1, \ldots, 16$)

Harmonic	Sin	Cos	Amplitude
0	0	−3.2415	3.2415
$\omega/4$	0.5278	−0.5708	0.7774
$\omega/2$	9.9741	7.6932	12.5964
$3/4\omega$	−0.0201	−1.0667	1.0669
ω	−30.2576	−0.3138	30.2592
$5/4\omega$	0.151	0.3043	0.3397
$3/2\omega$	0.2172	−0.0064	0.2173
$7/4\omega$	−0.1981	−0.2078	0.2871
2ω	−0.1384	0.3512	0.3775
$9/4\omega$	0.421	0.1979	0.4652
$5/2\omega$	−0.1315	−0.3162	0.3425
$11/4\omega$	−0.5769	0.1225	0.5898
3ω	0.5754	0.1795	0.6027
$13/4\omega$	1.7804	−2.4618	3.0382
$7/2\omega$	0.6258	−0.296	0.6923
$15/4\omega$	−0.038	−0.7144	0.7154
4ω	−0.0078	0.0539	0.0545

Table 3 Nonlinearity, Complexity index Θ and number *N* of terms in the Fourier series

Nonlinearity	Zero	Weak	Strong	Very strong (chaos)
Θ	1	1	>1	∞
N	1	>1	$\gg 1$	∞

Appendix

The mathematical model used for the considered beam of Leonhard Euler (1707–1783) with several transverse one-side non-propagating breathing cracks is based on the finite element model. According to the principle of Ademare Jean-Claude Barré de Saint-Vénant (1797–1886) the stress field is influenced only in the region adjacent to the crack. The element stiffness matrix, with the exception of the terms which represent the cracked element, may be regarded as unchanged under a certain limitation of the element size. The additional stress energy of a crack leads to a flexibility coefficient expressed by a stress intensity factor derived by means of the theorem of Carlo Alberto Castigliano (1847–1884) in the linear elastic regime.

The cracked element is shown in Fig. 3.

Neglecting shear action (Euler beam), the strain energy of an element without a crack can be obtained as

$$W^{(0)} = \frac{1}{2EI}\int_0^l (M+Pz)^2 dz = \frac{1}{2EI}(M^2 + P^2 l^3/3 + MPl^2), \tag{A1}$$

where E is the Young's modulus of the material constituting the finite element, $I = bh^3/12$ is the moment of inertia of its cross section, having base b and height h, and M and P are the generalized forces acting at the ends of the finite element of length l. The additional energy due to the crack is

$$W^{(1)} = b\int_0^a [[K_I^2(x) + K_{II}^2(x)]/E' + (1+\nu)K_{III}^2(x)/E]dx, \tag{A2}$$

where $E' = E$ for plane stress, $E' = E/(1+\nu)$ for plane strain and ν is the Poisson's ratio. $K_{I,II,III}$ are the stress intensity factors for opening, sliding and tearing-type crack, of depth a, respectively.

Taking into account only bending

$$W^{(1)} = b\int_0^a [[K_{IM}(x) + K_{IP}(x)]^2 + K_{IIP}^2(x)]/E' dx, \tag{A3}$$

with

$$K_{IM} = (6M/bh^2)\sqrt{\pi a}F_I(s)$$

$$K_{IP} = (3Pl/bh^2)\sqrt{\pi a}F_I(s),$$

$$K_{IIP} = (P/bh)\sqrt{\pi a}F_{II}(s) \tag{A4}$$

where $s = a/h$ and

$$F_I(s) = \sqrt{2/(\pi s)\tan(\pi s/2)} \times \{0.923 + 0.199[1-\sin(\pi s/2)^4]\}/\cos(\pi s/2)$$

$$F_{II}(s) = (3s - 2s^2)(1.122 - 0.561s + 0.085s^2 + 0.18s^3)/\sqrt{1-s}. \tag{A5}$$

The term $c_{ik}^{(0)}$ of the flexibility matrix $[C_e^{(0)}]$ for an element without crack can be written as

$$c_{ik}^{(0)} = \frac{\partial^2 W^{(0)}}{\partial P_i\, \partial P_k} \quad i,k = 1,2 \quad P_1 = P, P_2 = M. \tag{A6}$$

The term $c_{ik}^{(1)}$ of the additional flexibility matrix $[C_e^{(1)}]$ due to the crack can be obtained as

$$c_{ik}^{(1)} = \frac{\partial^2 W^{(1)}}{\partial P_i\, \partial P_k} \quad i,k = 1,2 \quad P_1 = P, P_2 = M. \tag{A7}$$

The term c_{ik} of the total flexibility matrix $[C_e]$ for the damaged element is

$$c_{ik} = c_{ik}^{(0)} + c_{ik}^{(1)}. \tag{A8}$$

From the equilibrium condition (Fig. 3)

$$(P_i\ M_i\ P_{i+1}\ M_{i+1})^T = [T](P_{i+1}\ M_{i+1})^T, \tag{A9}$$

where

$$[T] = \begin{bmatrix} -1 & -l & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}^T. \tag{A10}$$

Applying the theorem of Enrico Betti (1823–1892), the stiffness matrix of the undamaged element can be written as

$$[K_e] = [T][C_e^{(0)}]^{-1}[T]^T, \tag{A11}$$

or

$$[K_e] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}, \tag{A12}$$

while the stiffness matrix of the cracked element may be derived as

$$[K_{de}] = [T][C_e]^{-1}[T]^T, \tag{A13}$$

In order to evaluate the dynamic response of the cracked beam when acted upon by an applied force, it is supposed that the crack does not affect the mass matrix. Therefore, for a single element, the mass matrix can be formulated directly

$$[M_e] = [M_{de}] = \frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}, \tag{A14}$$

where m is the mass for unity length of the beam.

Assuming that the damping matrix $[D]$ is not affected by the crack, it can be calculated through the inversion of the modeshape matrix $[\phi]$ relative to the undamaged structure

$$[D] = ([\phi]^T)^{-1}[d][\phi]^{-1}, \tag{A15}$$

where $[d]$ is the following matrix:

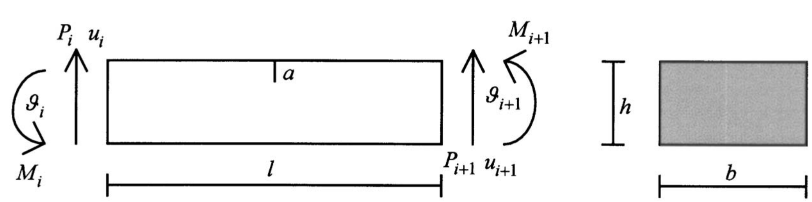

Fig. 3 Cracked element

$$[d] = 2\begin{bmatrix} \zeta_1\omega_1 M_1 & 0 & 0 & \dots & 0 \\ 0 & \zeta_2\omega_2 M_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \zeta_n\omega_n M_n \end{bmatrix}, \quad \text{(A16)}$$

in which ζ_i is the modal damping ratio, ω_i is the ith natural frequency, and M_i is the ith modal mass relative to the undamaged beam.

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