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The fracture mechanics of finite crack extension

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Abstract

This paper describes a modification to the traditional Griffith energy balance as used in linear elastic fracture mechanics (LEFM). The modification involves using a finite amount of crack extension (Δa) instead of an infinitesimal extension (da) when calculating the energy release rate. We propose to call this method finite fracture mechanics (FFM). This leads to a change in the Griffith equation for brittle fracture, introducing a new term $\Delta a/2$: we denote this length as L and assume that it is a material constant. This modification is extremely useful because it allows LEFM to be used to make predictions in two situations in which it is normally invalid: short cracks and notches. It is shown that accurate predictions can be made of both brittle fracture and fatigue behaviour for short cracks and notches in a range of different materials. The value of L can be expressed as a function of two other material constants: the fracture toughness K_c (or threshold ΔK_{th} in the case of fatigue) and an inherent strength parameter σ_0. For the particular cases of fatigue-limit prediction in metals and brittle fracture in ceramics, it is shown that σ_0 coincides directly with the ultimate tensile strength (or, in fatigue, the fatigue limit), as measured on plain, unnotched specimens. For brittle fracture in polymers and metals, in which larger amounts of plasticity precede fracture, the approach can still be used but σ_0 takes on a different value, higher than the plain-specimen strength, which can be found from experimental data. Predictions can be made very easily for any problem in which the stress intensity factor, K is known as a function of crack length. Furthermore, it is shown that the predictions of this method, FFM, are similar to those of a method known as the line method (LM) in which failure is predicted based on the average stress along a line drawn ahead of the crack or notch.

Keywords: Fracture mechanics; Short cracks; Notches; Brittle fracture; Fatigue

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1. Introduction

The science of linear elastic fracture mechanics (LEFM) has enjoyed great success in predicting the behaviour of bodies containing cracks, both in terms of brittle fracture and fatigue strength. For the purposes of this paper we define brittle fracture as any failure caused by crack propagation during the application of a monotonically increasing load; we define fatigue as gradual crack propagation during cyclic loading. However, LEFM is not able to predict the behaviour of short cracks or of notches. Short cracks (defined as cracks less than some critical length) have lower fracture strengths than predicted by LEFM and grow more quickly than expected under fatigue loading. Notches show similar behaviour to that of cracks if they are sharp (i.e. with a root radius less than some critical value) and if their opening angles are small. But otherwise notches tend to be less dangerous than cracks, in a manner which LEFM is unable to predict. These limitations are clearly very important when LEFM is used in practical applications to predict the effect of stress-concentrating features on the strength of components and structures.

In this paper we propose a modification to LEFM, introducing a material constant L, which has units of length. This constant arises through the use of the traditional energy balance approach with the added assumption of a finite (as opposed to infinitesimal) amount of crack extension. The aims of the paper are:

(i) To show how this modification, which we call finite fracture mechanics (FFM) widens the field of application of LEFM, allowing it to be used to predict the behaviour of short cracks and notches.
(ii) To examine the relationship between FFM and existing stress-based theories of the critical-distance type, especially the so-called line method (LM).
(iii) To compare predictions using FFM and LM with experimental data on brittle fracture and fatigue from short cracks and notches in various materials.

2. Development of the theory

The well-known prediction of brittle fracture developed by Griffith and Irwin, which is the basis of LEFM, proceeds as follows. We consider a straight, through-crack of length $2a$ in a flat plate whose width and length are large enough to be considered infinite, subjected to a remote tensile stress σ applied normal to the crack. The strain energy per unit thickness associated with the half-length a of the crack is W, where:

$$W = \frac{\sigma^2 a^2 \pi}{2E} \tag{1}$$

Normally, one considers the energy changes which occur when the crack length increases by an infinitesimal amount, $(\mathrm{d}a)$. The change in strain energy $\mathrm{d}W$ is equated to the energy needed for crack growth, $G_c(\mathrm{d}a)$, giving:

$$G_c = \frac{\sigma^2 a \pi}{E} \tag{2}$$

The stress in Eq. (2) is the predicted fracture stress, σ_f, thus:

$$\sigma_f = \sqrt{\frac{G_c E}{\pi a}} = \frac{K_c}{\sqrt{\pi a}} \tag{3}$$

Here E is equal to Young's modulus if plane stress conditions prevail, and equal to Young's modulus divided by $(1 - \nu^2)$, ν being Poisson's ratio, under plane strain. Equating $G_c E$ to K_c^2 is a matter of definition. This equation can be used to predict unstable crack extension under monotonic loading (i.e. brittle fracture) when K_c is the fracture toughness. It can also be used to predict the threshold conditions for fatigue crack propagation, when σ_f becomes the range of cyclic stress at the fatigue limit of the specimen containing the crack, and K_c becomes the fatigue crack propagation threshold for the material (normally written ΔK_{th}).

3. A modified energy balance with finite crack extension

Consider a modification to the above in which the amount of crack extension is not infinitesimal (da) but rather a finite value, Δa. Possible physical reasons for this will be considered later: Δa is assumed to be constant for a given material and given fracture process (e.g. brittle fracture or fatigue). The associated change in strain energy, ΔW, is:

$$\Delta W = \int_a^{a+\Delta a} \frac{\sigma^2 \pi}{E} a \,\mathrm{d}a \tag{4}$$

$$= \frac{\sigma^2 \pi}{2E}\left[2a\Delta a + \Delta a^2\right] \tag{5}$$

Equating this to $G_c \Delta a$ gives a new form for the fracture stress:

$$\sigma_f = \sqrt{\frac{G_c E}{\pi(a + \Delta a/2)}} = \frac{K_c}{\sqrt{\pi(a + \Delta a/2)}} \tag{6}$$

When a is large compared to Δa, this equation reverts to the normal LEFM prediction for long cracks (Eq. (3)). But as a decreases the fracture stress becomes lower than predicted by LEFM, tending to a constant value as a approaches zero. We will call this the 'inherent strength' of the material, σ_0. Thus:

$$\sigma_0 = \frac{K_c}{\sqrt{\pi(\Delta a/2)}} \tag{7}$$

This inherent strength may or may not be equal to the strength of plain (i.e. uncracked) specimens, which is the ultimate tensile strength (UTS) in monotonic loading and the fatigue limit in cyclic loading. See below for further discussion on this point. We define a material constant, L, which is equal to (Δa/2) and can be found if the material's inherent strength and toughness are known:

$$L = \frac{1}{\pi}\left(\frac{K_c}{\sigma_0}\right)^2 \tag{8}$$

Fig. 1 shows the form of Eq. (6), assuming that σ_0 = UTS, using some typical material parameters; also shown is the LEFM prediction and a horizontal line representing the UTS. Equations of exactly the same form as Eq. (6) have previously been proposed for predicting the behaviour of short cracks, by Suo et al. [1] for brittle fracture and (independently) by ElHaddad et al. [2] for fatigue. These workers proposed the equation simply as an empirical law which could be shown to fit satisfactorily to experimental data. The same equation can also be derived in a different way [3,4], using the so-called line method (LM) in which failure is assumed to occur if the inherent strength is equal to the average stress on a line drawn in the direction of crack extension, starting at the crack tip and extending a length equal to 2L. The proof of this is as follows: for a central through-crack in an infinite plate the stress in the crack-opening direction, σ_{yy}, along this line is given as a function of distance from the crack tip, r, by the equation of Westergaard [5]:

$$\sigma_{yy} = \sigma \frac{1}{\sqrt{1 - \left(\frac{a}{a+r}\right)^2}} \tag{9}$$

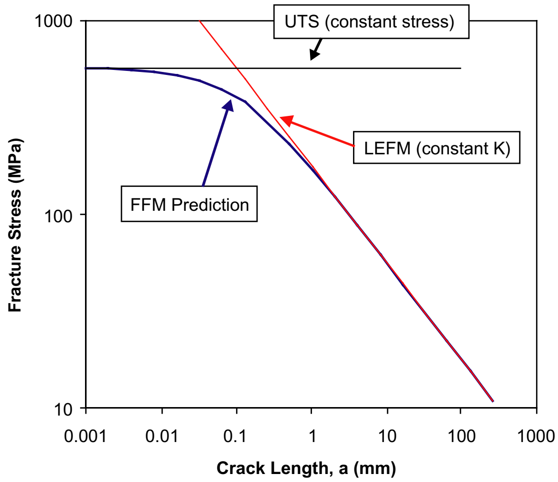

Fig. 1. Prediction using finite fracture mechanics (FFM) of the variation of fracture stress (σ_f) with crack length (a) using some arbitrary material constants ($K_c = 10\,\text{MPa}\,(\text{m})^{1/2}$, $L = 0.1\,\text{mm}$). Also shown are lines corresponding to the plain-specimen strength (UTS) and the prediction obtained using normal LEFM theory.

The average value of σ_{yy} over the length $r = 0$ to $2L$ is thus:

$$\sigma_{\text{av}} = \frac{1}{2L}\int_0^{2L} \sigma \frac{1}{\sqrt{1-\left(\frac{a}{a+r}\right)^2}}\,\mathrm{d}r \tag{10}$$

$$= \sigma\sqrt{\frac{a+L}{L}} \tag{11}$$

According to the LM, $\sigma_{\text{av}} = \sigma_0$ when $\sigma = \sigma_f$ therefore, using Eq. (8) gives:

$$\sigma_f = \frac{K_c}{\sqrt{\pi(a+L)}} \tag{12}$$

This is the same as Eq. (6), showing that LM and FFM give exactly the same result with the same value of L. The LM is one of a number of critical-distance methods which use some features of the stress field ahead of a crack or notch (see elsewhere [4] for more details of these methods as applied to fatigue). Because of the similarity of this equation when derived in different ways, there already exists a body of literature to show that Eq. (6) is capable of predicting the experimental data, especially in fatigue and in the fracture of brittle ceramics. A comparison of various theories with experimental data will be carried out below. The above derivation was carried out only for the simple geometry of a through-crack in an infinite plate in tension. The necessary modifications for considering other geometries (and the issues that arise in the process) will be considered below.

4. Extending the theory to consider notches

Two parameters affect the behaviour of notches in comparison to that of cracks: root radius and notch angle. Here we will consider the effect of root radius for a notch of zero angle (i.e. a U-shaped notch), with length a_n and root radius ρ. Fig. 2 shows a through-thickness edge notch, though in fact the derivation will apply to any shape (e.g. an elliptical cavity) provided a_n is much smaller than the dimensions of the body

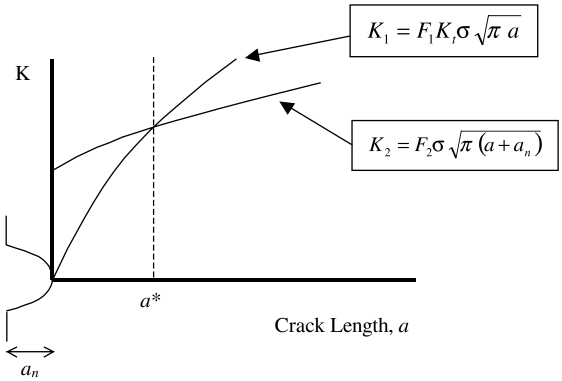

Fig. 2. Approximate solutions for K for a crack growing from a notch.

itself. The loading again takes the form of a remote, normal, tensile stress σ. Fig. 2 shows how the stress intensity, K, increases for a crack growing from the root of the notch. The form of this increase is complex and difficult to represent analytically, so we will use a simplified form which is commonly used in notch/crack analysis (for example Yates and Brown [6]). When the crack is relatively small its stress intensity is approximately given by K_1, where:

$$K_1 = F_1 K_t \sigma \sqrt{\pi a} \tag{13}$$

Here K_t is the elastic stress concentration factor of the notch (equal to the maximum stress at the notch root divided by σ) and F_1 is a constant which depends on the geometry of the notch and crack. When the crack is relatively large its stress intensity is given by K_2, where:

$$K_2 = F_2 \sigma \sqrt{\pi (a + a_n)} \tag{14}$$

Here F_2 is the geometry factor for a crack of total length ($a + a_n$). The two solutions cross at $a = a^*$, where:

$$a^* = a_n \frac{F_2^2}{(F_1^2 K_t^2 - F_2^2)} \tag{15}$$

As before we consider the energy changes consequent on a finite amount of crack growth, $\Delta a = 2L$. There are two possible cases:

Case 1: $2L < a^*$

In this case only Eq. (13) is needed:

$$\sigma_\mathrm{f} = \frac{K_\mathrm{c}}{F_1 K_t \sqrt{\pi L}} \tag{16}$$

The full effect of the stress concentration factor ($\sigma_\mathrm{f} = \mathrm{UTS}/K_t$) is experienced if $\sigma_0 = F_1 \cdot \mathrm{UTS}$. See below for a discussion about this point.

Case 2: $2L > a^*$

In this case the change in strain energy is:

$$\Delta W = \int_0^{a^*} \frac{K_1^2}{E} \mathrm{d}a + \int_{a^*}^{2L} \frac{K_2^2}{E} \mathrm{d}a \tag{17}$$

$$= \frac{\sigma^2 \pi}{E} \left[a_n (2L) - \frac{F_2^2}{2} \frac{a_n^2}{(F_1^2 K_t^2 - F_2^2)} + \frac{(2L)^2}{2} \right] \tag{18}$$

Equating ΔW to $G_c(2L)$ gives a prediction for σ_f for the notch:

$$\sigma_f = \frac{1}{F_2}\frac{K_c}{\sqrt{\pi Q}} \tag{19}$$

where $Q = a_n - \frac{F_2^2}{2}\frac{a_n^2}{(F_1^2K_t^2 - F_2^2)2L} + L$.

The parameter Q has three terms. The first term, a_n, dominates in cases of long, sharp cracks ($a_n \gg L$ and K_t = infinity), when Eq. (19) is the same as Eq. (3) except for the shape factor F_2. The second term modifies the equation to account for notches; the third term, L, controls the size effect, giving a reduced strength which tends to σ_0 as the length of the crack or notch tends to zero.

Fig. 3 shows the form of Eqs. (16) and (19) for an example where $a_n/L = 10$, plotting σ_f/σ_0 as a function of K_t. Eq. (16) is valid at low K_t and Eq. (19) at high K_t, so we will refer to these as the 'blunt' and 'sharp' solutions respectively; they coincide when $a^* = 2L$. At high stress concentration factors the solution becomes horizontal, being asymptotic to the result for a sharp crack of the same length. Similar behaviour in the experimental data on fatigue limits for notched specimens was found by Frost et al. [7] and by Smith and Miller [8]. The latter workers proposed that predictions could be made using two equations, for blunt and sharp notches respectively. Their equation for blunt notches is identical to the one used here (Eq. 16). They assumed that sharp notches were exactly crack-like, giving a horizontal line on the figure to which our Eq. (19) is asymptotic.

Fig. 4 shows how the result changes with normalised notch length (plotting only the valid parts of the curves in each case): for small notches the strength tends to a constant value at all K_t, which approaches unity as a_n/L approaches zero. Fig. 5 shows the results expressed in a different way, plotting the measured K value at failure (denoted K_b) normalised by the long-crack value K_c. K_b is defined as:

$$K_b = F_2\sigma_f\sqrt{\pi a_n} \tag{20}$$

It is thus the value of K_c that would be measured in an experiment using a notched specimen, assuming that the notch was the same as a crack. In Fig. 5 we plot the result against the notch root radius, ρ, normalised by L. To do this it is necessary to assume some relationship for K_t: we have taken the one for elliptical notches, which is also reasonably accurate for many other notch shapes:

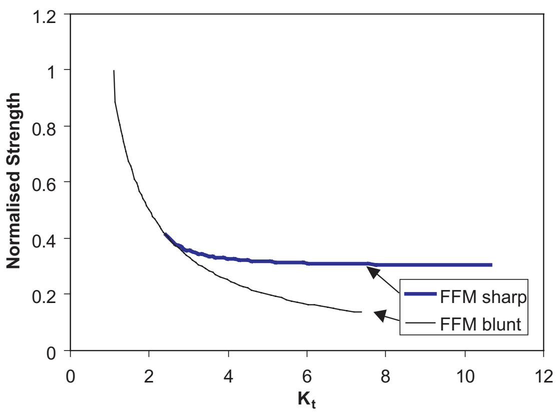

Fig. 3. Predictions of normalised strength (σ_f/σ_0) as a function of K_t for a notch with $a_n = 10L$, showing the 'blunt' and 'sharp' solutions: Eqs. (16) and (19) respectively.

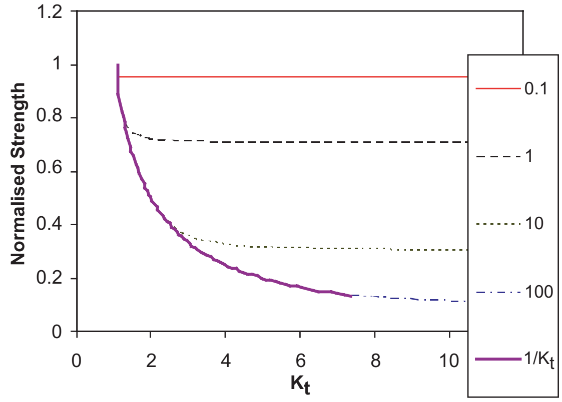

Fig. 4. Normalised strength versus K_t (Eq. (15)) for various values of a_n/L. The 'blunt' solution (Eq. (16)) is identical for all cases (here denoted $1/K_t$).

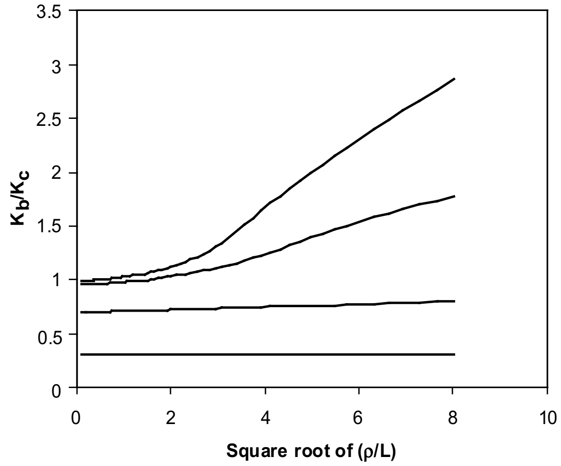

Fig. 5. Predictions for K_b/K_c as a function of ρ/L for various values of a_n/L. Reading from the top downwards, the lines represent $a_n/L = 100$; 10; 1; 0.1.

$$K_t = 1 + 2\sqrt{\frac{a_n}{\rho}} \tag{21}$$

The figure shows that, for macroscopic notches ($a_n/L \gg 1$), the notch behaves like a long crack ($K_b/K_c = 1$) up to some critical value of ρ, which varies with notch size but is of the same order of magnitude as the material parameter L. Smaller notches all show crack-like behaviour but also display short-crack effects, making $K_b < K_c$. A large amount of experimental data exists in the literature, showing behaviour of this type for brittle fracture and fatigue; some of this data will be presented below.

Eqs. (16) and (19) reduce to very simple forms in a case of particular interest: a long, sharp, though-thickness edge notch ($a_n \gg L$; $a_n \gg \rho$; $F_1 = F_2 = 1.12$). In this case the value of K_t given in Eq. (21) should be increased by a factor 1.12. The results are, for the sharp notch solution:

$$\frac{K_b}{K_c} = \sqrt{\frac{1}{1 - \frac{\rho}{20.08L}}} \quad (22)$$

and for the blunt notch solution:

$$\frac{K_b}{K_c} = \frac{1}{2.24}\sqrt{\frac{\rho}{L}} \quad (23)$$

Exactly the same result occurs for the case of an elliptical hole; in that case $F_1 = F_2 = 1$ but any change this causes is cancelled out by the change in K_t.

It should be remembered that all these solutions for notches are approximate, based on the 'two curves' simplification (Fig. 2), which will tend to give a value of strain energy which is rather larger than the true value, and will thus tend to underpredict the strength. A more accurate analysis can be carried out using K values taken from the literature [9] for particular notch shapes, as will be shown below.

5. Comparison with experimental data

This section presents experimental data taken from various sources in the literature, on the behaviour of short cracks and notches in fatigue and brittle fracture. Predictions will be made using FFM and also the LM, for comparison. Fig. 6 shows typical data on the fracture strength of a ceramic material—silicon carbide—tested by Kimoto et al. [10]. The FFM theory (Eq. (6)) predicts the results very well, over the whole range of crack lengths from long cracks (which conform to standard LEFM predictions) to very short cracks which have no significant effect on specimen strength. The material had a plain-specimen tensile strength (UTS) of 620 MPa and a fracture toughness (K_c) of 3.7 MPa(m)$^{1/2}$, giving a value of L (from Eq. (8)) of 0.011 mm. Many workers have measured the fatigue strength of specimens containing short cracks and have found similar effects. It is generally accepted that the equation of ElHaddad et al. [2] can be used to predict this data, and we noted above that the ElHaddad equation is identical to our Eq. (6) (ElHaddad's constant a_0 being equal to our L). Taylor and O'Donnell [11] conducted a survey of data

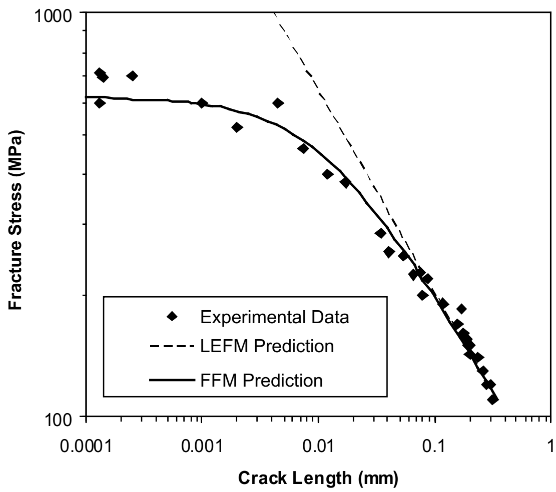

Fig. 6. Experimental data on the strength of silicon carbide [10], showing the LEFM prediction and the FFM prediction (the LM prediction is identical to FFM in this case).

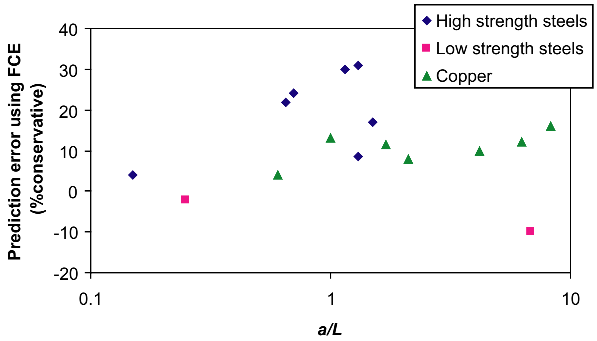

Fig. 7. Accuracy of FFM predictions applied to the fatigue limits of cracked specimens (from Taylor and O'Donnell [11]).

from various materials, the results of which are presented in Fig. 7, which shows the prediction error arising from using this equation, as a function of normalised crack length. It can be seen that the equation is very successful: errors rarely exceed 15%, which is very good considering the difficulties of making these measurements experimentally. Most errors are conservative (i.e. the predicted fatigue strength is higher than the experimental value): there are some errors in the range 20–30% for crack lengths of the order of L. Looking again at Fig. 1, we realise that this experimental data will lie either close to the prediction line or else in the region above the prediction line but below the boundary formed by the two straight lines (which correspond to the constant-K and constant-stress solutions respectively). For points in this region the maximum possible error (i.e. the maximum thickness of the region) is a factor of 1.4, occurring at $a = L$. Some workers [12] have suggested that low-strength steels follow Eq. (6), whilst higher strength steels display relatively higher fatigue limits, approaching the two straight lines. This issue will be discussed again below.

Fig. 8 shows some data due to Tsuji et al. [13] on the measured fracture toughness of ceramic specimens containing cracks and notches, as a function of root radius. The material is alumina (material properties: $K_c = 3.83\,\text{MPa}\,(\text{m})^{1/2}$, UTS = 297 MPa, giving $L = 0.052$ mm). The LM and FFM theories both give reasonable predictions, with the FFM (blunt and sharp solutions) forming a lower bound to the data and giving a slight underestimate of strength at high ρ. As noted above this is due to the simplification used in estimating K_1 and K_2, which tends to give an overestimate of the strain energy. The FFM prediction can be made more

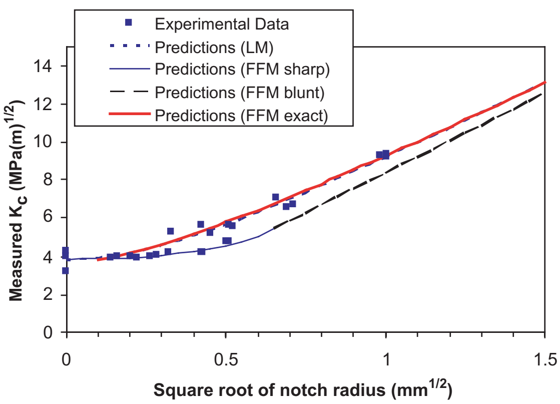

Fig. 8. Experimental data showing measured K_c as a function of notch root radius for alumina. Predictions using FFM (blunt, sharp and exact solutions) and also using LM.

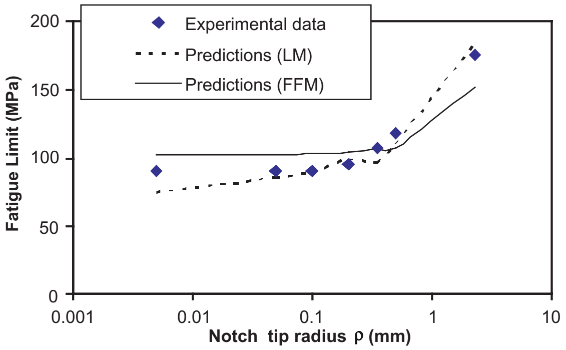

Fig. 9. Data [8] and predictions on the fatigue limits of notched specimens; here the sharp-FFM solution applies for all except the largest ρ value.

accurately in a particular case by using the appropriate F factors for stress intensity, taken from Murakami [9]. We carried out this analysis for the data on Fig. 8, giving the prediction line labelled "FFM exact". This shows that FFM and the LM give almost identical predictions, though there are some slight differences. This exact FFM calculation can only be done knowing the F factor as a function of crack length for each individual specimen geometry, so we will continue to use the simplified, general form below. Fig. 9 shows data from Smith and Miller [8] on the fatigue strength of notched specimens. The material is a mild steel (0.15% carbon) with a plain-specimen fatigue limit of 420 MPa and a crack propagation threshold of 12.8 MPa(m)$^{1/2}$, giving $L = 0.30$ mm. The comparison between data, LM and FFM predictions is very good.

Up to now we have been able to make accurate predictions using Eq. (8) to calculate L, assuming that the inherent strength σ_0, is equal to the plain-specimen strength (i.e. the UTS or fatigue limit of the material), incorporating crack shape effects through the F factor. We have found this to be successful in predicting many sets of data on the fatigue strengths of metals and the fracture strengths of ceramics. This is very convenient because it means that no new constants are required in order to use the theory. However when we considered brittle fracture in polymers and metals, we found that, although accurate predictions could be made, it was necessary to use a different value of L. Figs. 10 and 11 illustrate this for data on a polymer

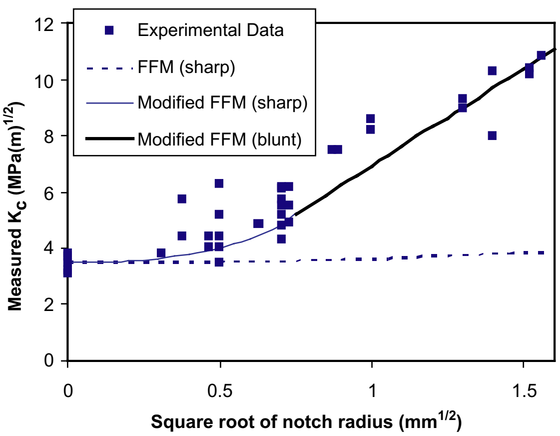

Fig. 10. Data for brittle fracture in polycarbonate [13]. "FFM (sharp)" refers to the FFM prediction using σ_0 = UTS; in the "modified FFM" we used a lower value of L to obtain a best fit to the data.

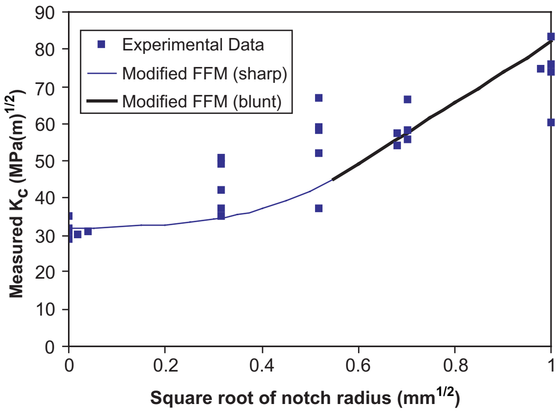

Fig. 11. Data and modified FFM predictions for brittle fracture of steel at low temperature.

Table 1
Values of constants used in predicting brittle fracture in polycarbonate and steel

	Polycarbonate	Steel
Fracture toughness K_c (MPa(m)$^{1/2}$	3.47	31.8
Plain-specimen UTS (MPa)	70.2	810
L, found using UTS (mm)	0.78	0.49
L', found by best-fit (mm)	0.05	0.03
Value of σ_0 found using L' and K_c (MPa)	277	3275
σ_0/UTS	3.9	4
L'/L	15.6	16

(polycarbonate) and a metal (mild steel tested at −170 °C), both reported by Tsuji et al. [13]. Both the original and modified predictions are shown in Fig. 10, indicating that the value of L calculated using the UTS was too large. Accurate predictions were obtained using a lower value, L', which implies a value of σ_0 higher than the UTS by a factor of about 4 in both cases. The same modification was needed when using LM. Table 1 lists the various material constants. This is less satisfactory from a prediction point of view, as the value of L' can only be known by finding a best fit to the experimental data. This implies that data is required for two different notches—ideally a crack and a relatively blunt notch. Even so the predictive capacity of the theory is still very high.

6. Discussion

We have shown that an energy balance approach using the assumption of a finite crack extension causes a modification to traditional LEFM which allows predictions to be made of the behaviour of short cracks and of notches. In both cases, deviations from LEFM occur when the relevant physical size (a or ρ respectively) is similar to, or smaller than, some value L. The fact that L takes the same value in both cases, and the same FFM theory can be used to predict both, means that the two phenomena can be viewed in a unified manner. To our knowledge, no similar theory has previously been proposed to solve these problems. Novozhilov [14] proposed that a crack propagates not smoothly but in discrete 'quanta'; however for him the quantum of advance was an individual atomic bond. By applying this approach, removing the hypothesis of fracture quantum equal to the atomic size, Carpinteri and Pugno [15] were able to evaluate the strength of structural elements containing re-entrant corners. Seweryn [16] also proposed an energy balance approach using a finite amount of crack extension, to consider the propagation of a long crack under mixed-mode loading. Such cracks develop kinked extensions; Seweryn proposed both LM and FFM-type models to define the conditions for propagation of this kink. He deduced a value for the kink length which is similar to our $2L$, assuming that σ_0 = UTS. The same approach was also proposed for sharp, V-shaped notches [17] but in this case it was concluded that the method of analysis was too difficult to ensure accurate predictions. The reason for this conclusion was that a different method was used to estimate ΔW, based on local notch-root stresses and crack openings. This method is difficult to use due to the lack of an accurate function for the crack openings for crack-notch combinations. This point is discussed further in Appendix A, where our own and Seweryn's methods are compared for two particular crack geometries. A great advantage of the present method is that predictions can be made very easily, for any problem for which the appropriate equation for K is already known (e.g. from Murakami [9]).

Another related (but distinctly different) model of notch behaviour involves placing a pre-existing crack at the notch root, using fracture mechanics to predict the conditions for failure of this notch/crack combination (e.g. [17–19]). At first sight this appears similar to the present argument, but there are two important differences. Firstly, the idea breaks down if we sharpen the notch to the point at which it becomes a crack, because if a small crack is to exist at the tip of this crack then what we have is simply a longer crack so there is no reason why another small crack should not be added to it, and so on ad infinitum. The problem does not arise in the case of FFM; we propose that the crack advances discontinuously, in quanta of length $2L$. Secondly, in predicting the growth of this small crack it is generally assumed that LEFM applies, which it certainly does not because the length of the crack is too small.

As with traditional LEFM, we have implicitly assumed conditions of nominal elastic behaviour: i.e. the size of plastic zones at the notches and cracks is much less than the specimen dimensions (though not, in the case of short cracks, necessarily smaller than the crack length). It is possible that the same FFM approach might be extended to elastic–plastic fracture mechanics, providing a modification to the J integral parameter, but this development is beyond the scope of the present paper.

In two cases studied—the fatigue limits of metals and the fracture of ceramics—the plastic zones (taken to include any zones of non-linear or irreversible deformation in the case of non-metals) are very small indeed, certainly smaller than L. When we considered problems where the plastic zones are somewhat larger (brittle fracture in polymers and metals), the same theory could be used but now the inherent material strength σ_0 was found to be larger than the measured UTS. The same effect occurs when using LM, and Kinloch and Williams [20] noted that a similar correction was needed (increasing the UTS by a factor of about 3) when using another type of critical distance method, this one being based on the stress value at a given distance from the notch. This problem arises because the failure of plain specimens occurs by a different mechanism from the failure of notched and cracked specimens. General yielding/crazing occurs throughout the remaining section and final failure is no longer due to crack propagation. In this case the UTS ceases to be a useful parameter in our predictions. The inherent strength (which in these cases is invariably higher than the UTS) can perhaps be thought of as the strength which the material *would have had* if the other failure mechanism (yielding/crazing) had not occurred. However this is probably not a useful line of thought: the inherent strength probably has no physical meaning. The two physically meaningful parameters are K_c (which determines, along with E, the amount of energy needed for crack propagation) and the length constant L.

In this paper we do not intend to make an extended discussion on the physical meaning of L. Suffice it to say that many theories developed over the years have proposed the use of some material length parameter in fracture studies, and even in stress analysis. On the other hand, while the physical meaning of the proposed discrete approach remains partially unclear at micro-, meso- and macro-scale, it becomes very clear at nanoscale, where to fit the numerical and experimental results, a finite crack extension equal to the distance between two adjacent chemical bonds has to be considered [21,22].

The first person to propose a material length parameter was Neuber [23], who suggested it first as a basic tenet of stress analysis and subsequently used it extensively, especially in fatigue studies. Neuber's method was to average the stress over a distance—the approach which we now call LM—which he assumed was related to the size of microstructural features in the material (the 'microstructural support length'). Our FFM theory, when expressed in physical terms, assumes that failure proceeds by the entire fracture of one of these microstructural units. In practice the size of these units is probably related either to the spacing of inherent defects in the material (e.g. pores, inclusions) or to the spacing of physical barriers to crack growth (e.g. grain boundaries). The relevant feature will vary from material to material, since different mechanisms are at work to facilitate or prevent crack growth. In notch studies, cracks are often seen to initiate but then to stop growing—to become 'non-propagating' if the applied stress is below the critical value for failure. Taylor [24] has noted that the length of these non-propagating cracks in fatigue is similar to $2L$, and has proposed that the LM specifies the conditions for continued propagation of these cracks beyond this critical length.

The results of the present paper also suggest a link between FFM and stress-based critical distance methods such as the LM. We showed that the predictions of the two theories are very similar, using the same distance, $2L$, for the crack extension as for the distance over which stresses are averaged in the LM. Traditional LEFM parameters can be derived using either an energy balance approach or a stress-field approach, and it seems that the same is true for this modification of LEFM. An analogy may be drawn between the two commonly used theories of yielding: Von Mises (energy-based) and Tresca (stress-based). In fact, when we look more carefully we find that, just as with Tresca and Von Mises, there are some differences between FFM and LM. The equivalence of short-crack predictions (Eq. (6)) only holds true for the centre-cracked plate, for which the geometry factor F is unity. If we include this factor the equation becomes (writing L instead of $\Delta a/2$):

$$\sigma_{\mathrm{f}} = \frac{1}{F}\frac{K_{\mathrm{c}}}{\sqrt{\pi(a+L)}} \tag{24}$$

The same factor appears as F_2 in Eq. (19) for notches, and will persist in the equation for inherent strength:

$$\sigma_0 = \frac{1}{F}\frac{K_{\mathrm{c}}}{\sqrt{\pi L}} \tag{25}$$

This appears to create a problem because it implies that the strength of crack-free material is no longer a constant, but depends on the shape of the crack, even though there is no crack present! This only serves to illustrate the point made above, that σ_0 has no physical meaning. If we plot Eq. (24) for a series of cracks with different shapes, a series of parallel curves is created (Fig. 12), giving different values of σ_0. This does not occur with LM, for which the various different curves all tend to the same value at zero crack length.

We noted earlier that in some materials the actual UTS is lower than σ_0 due to failure by general yielding, crazing, etc. Another reason why the UTS may be lower is that the material will contain inherent defects, so if a is small enough then failure will occur not from the crack which we introduced but from one of these defects. Alternatively, small cracks may be initiated by the loading as happens, for example, in fatigue and in the formation of crazes in PMMA. In that case the UTS (or fatigue limit) will be lower than any of the σ_0 values here, and will intersect with our prediction line at a point corresponding to the size and shape of the defects concerned. In practice F varies from 1.12 (for a through edge-crack) to around 0.7 for a buried, circular crack. Lower values of F are unusual; F can be larger than 1.12 but this is usually due to finite specimen size (i.e. finite width in the crack-growth direction). In this case F will also be expected to change as the crack grows, an effect which, if incorporated into the predictions, will tend to reduce σ_{f}. In Fig. 12 the UTS line corresponds to failure of a crack of length 0.2 mm and $F = 0.8$.

Whatever the reason for the particular UTS value, it will provide a cut-off point for the various FFM prediction lines, since the strength cannot be higher than the UTS. For this reason it should be possible

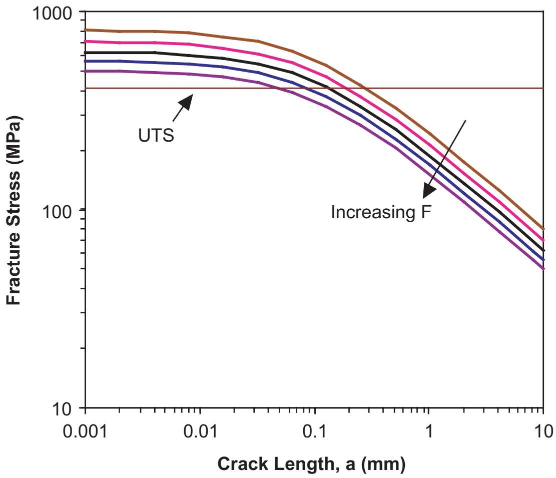

Fig. 12. Prediction lines (using the same material constants as Fig. 1) for cracks with different values of the shape factor, F. The values used (reading from top to bottom) were: 0.7; 0.8; 0.9; 1.0 and 1.12. The horizontal line indicates the UTS of the material.

to test these predictions against experimental data. We should find that for short cracks, σ_f varies with crack shape, and that a cut-off occurs at the UTS as described above. In practice this is difficult because the differences are relatively small and effectively hidden in the scatter, so we were not able to find a set of data which was sufficiently accurate and extensive to test this point. The data on short cracks in ceramics (Fig. 6) was reported in terms of an 'effective crack length', which is defined as the length which the cracks would have had if their F factor had been unity. This is common practice in this field, but may tend to hide any effects of crack shape.

If the F factor is small, and if σ_0 is much greater than the UTS, then the curved portion of the prediction line below the UTS will be relatively small. In practice if σ_0 is much greater than twice the UTS it turns out that the predictions lie very close to the two straight lines in Fig. 1 (constant-stress and constant-K). This may be the explanation for the data on short fatigue cracks (Fig. 7) discussed above.

7. Conclusions

(1) The strength of bodies containing short cracks and notches cannot normally be predicted using LEFM, but accurate predictions become possible if the Griffith energy balance is modified, assuming a finite amount of crack extension, Δa, which is a material constant.
(2) Predictions can be made both of brittle fracture under monotonic loading and of high-cycle fatigue failure. Predictions are of good accuracy for a wide range of materials, including metals, polymers and ceramics.
(3) The appropriate amount of crack extension can be calculated as a function of two other material constants. The first is a limiting stress intensity: the fracture toughness in brittle fracture and the propagation threshold in fatigue; the second is an inherent strength σ_0 which in some cases is equal to the plain-specimen strength and in other cases takes a higher value.
(4) Using a method of integration of the strain energy release rate, it is possible to make the necessary calculations very easily for any problem in which the stress intensity, K, is known as a function of crack length.

(5) This method, which we call finite fracture mechanics (FFM) can be shown to give predictions similar to those of a method based on averaging stresses along a line ahead of the crack or notch (called the LM). The length of this line can be shown to be the same as the amount of finite crack extension.

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Appendix A. FFM predictions using the method of Seweryn, compared to our method of strain energy release rate integration

Seweryn and Lukaszewicz [17] proposed a discrete energy release failure criterion for crack propagation, which is essentially the same as the FFM theory proposed in this paper. However, he computed the energy release during the crack advance in a different way, using Clapeyron's theorem, as one half of the integral of the product of the stress field *before* crack advance times the displacement between crack lips *after* crack advance. In order to have the exact result one needs to know the stress and displacement field exactly. Using the asymptotic field in the crack tip vicinity causes an approximation in the results which becomes more important for larger crack extensions. This consideration forced Seweryn to conclude that the discrete energy release failure criterion is too complex to be applied. Here we will show that Seweryn's method gives the same prediction as our own, strain-energy-release-rate method, for the case of a central through crack under remote tension.

A.1. Seweryn's method

Let us consider a central through crack (Fig. 13). The remote tensile stress is σ, directed along y. Suppose that the crack advances (symmetrically) by Δ. The strain energy release can be computed applying

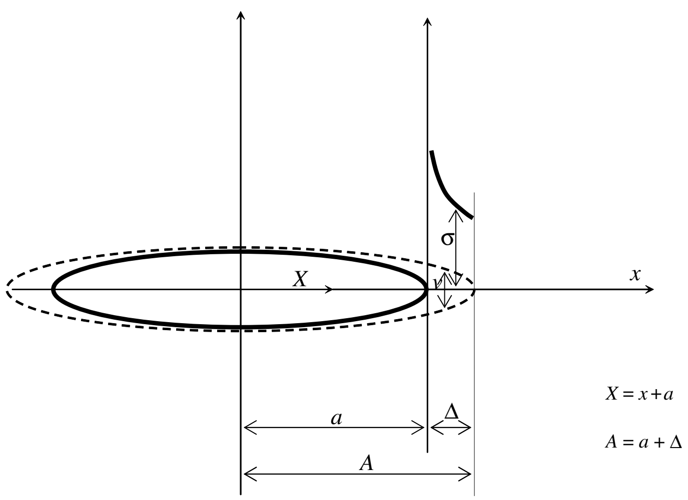

Fig. 13. Central crack under remote tension.

Clapeyron's theorem to the closure work, therefore considering the stress in the final configuration (before crack advance) and the displacement in the initial one (after crack advance). Hence, half of the work ($\Delta\phi$) done to close the crack lips for a Δ-long segment is given by

$$\Delta\phi = \frac{1}{2}\int_{a}^{a+\Delta} \sigma(X)v(X)\,\mathrm{d}X \tag{A.1}$$

where $\sigma(x)$ is the exact stress field (Westergaard's solution) ahead of the crack tip:

$$\sigma(x) = \frac{a+x}{\sqrt{x(x+2a)}}\sigma, \quad x \geqslant 0 \tag{A.2}$$

and $v(x)$ is the exact displacement field (distance between crack faces):

$$v(X) = \frac{4\sigma}{E}\sqrt{A^2 - X^2}, \quad X \leqslant A \tag{A.3}$$

The displacement function can be computed applying Castigliano's theorem and knowing the stress intensity factor solution for a pair of forces applied on the crack lips. Expressing the stress field as a function of X, we get:

$$\sigma(X) = \frac{X}{\sqrt{X^2 - a^2}}\sigma, \quad X \geqslant a \tag{A.4}$$

Substituting Eq. (A.4) into (A.3), we obtain:

$$\begin{aligned}
\Delta\phi &= \frac{2\sigma^2}{E}\int_{a}^{A} X\sqrt{\frac{A^2 - X^2}{X^2 - a^2}}\mathrm{d}X \\
&= \frac{2\sigma^2}{E}\left[2\sqrt{(A^2 - X^2)(X^2 - a^2)} - (A^2 - a^2)\arctan\frac{(A^2 - X^2) - (X^2 - a^2)}{2\sqrt{(A^2 - X^2)(X^2 - a^2)}}\right]_{X=a}^{X=A} \\
&= \frac{\sigma^2\pi}{2E}(A^2 - a^2) = \frac{\sigma^2\pi}{2E}\Delta(2a + \Delta)
\end{aligned} \tag{A.5}$$

The closure work equals the strain energy released during the discrete crack advance.

A.2. The method of strain energy release rate integration

The same result can be obtained by integration of the strain energy release rate, since:

$$G = \left(\frac{\mathrm{d}\phi}{\mathrm{d}a}\right)_{\text{fixedload}} = \frac{K_{\mathrm{I}}^2}{E} \tag{A.6}$$

As well known, $K_{\mathrm{I}} = \sigma\sqrt{\pi a}$, therefore:

$$\Delta\phi = \int_{a}^{a+\Delta}\frac{K_{\mathrm{I}}^2(a)}{E}\mathrm{d}a = \frac{\sigma^2\pi}{2E}\Delta(2a + \Delta) \tag{A.7}$$

which coincides with (A.5), thus showing that the two methods give the same result.

At propagation:

$$\Delta\phi = G_{\mathrm{c}}\Delta = \frac{K_{\mathrm{Ic}}^2}{E}\Delta \tag{A.8}$$

For $a = 0$, which implies $\sigma = \sigma_u$ (the UTS), and equating (A.7) and (A.8), we find the value of $\varDelta$:

$$\varDelta = \frac{2}{\pi}\left(\frac{K_{Ic}}{\sigma_u}\right)^2 \tag{A.9}$$

Thus $\varDelta$ has the same value as $2L$ as defined in this paper. Let us now consider an edge through crack under remote tension of length a. In this case the stress intensity factor is given by

$$K_I = 1.12\sigma\sqrt{\pi a} \tag{A.10}$$

Wishing to compute the strain energy release for a crack advance $\varDelta$ starting from $a = 0$, we can use Eq. (A.7) that yields:

$$\Delta\phi = \int_0^{\varDelta} \frac{K_I^2(a)}{E}\mathrm{d}a = 1.12^2\frac{\sigma^2\pi}{2E}\varDelta^2 \tag{A.11}$$

If, now, we want to recover, for crack propagation, the result $\sigma = \sigma_u$ for $a = 0$, it must be:

$$\Delta\phi = G_c\varDelta = \frac{K_{Ic}^2}{E}\varDelta = 1.12^2\frac{\sigma_u^2\pi}{2E}\varDelta^2 \tag{A.12}$$

that is:

$$\varDelta = \frac{2}{\pi}\left(\frac{K_{Ic}}{1.12\sigma_u}\right)^2 = 0.508\left(\frac{K_{Ic}}{\sigma_u}\right)^2 \tag{A.13}$$

Observe that, if we had considered $\varDelta$ a material parameter as given by Eq. (A.9), the discrete strain energy release fracture criterion would have provided a failure stress equal to 1/1.12 times σ_u. Seweryn performed the same computations but using Clapeyron's theorem. For this purpose, he used (i) the stress field $\sigma(x) = \sigma$ = constant before crack advance (which is exact) and (ii) the first order term of the displacement field:

$$v(x) = \frac{4}{E}\sqrt{\frac{2x}{\pi}}(K_I)_{a=\varDelta} \tag{A.14}$$

Applying Eq. (A.1) provides:

$$\Delta\phi = \frac{1}{2}\int_0^{\varDelta} \sigma(x)v(x)\mathrm{d}x = \frac{4.48\sqrt{2}(\sigma\varDelta)^2}{3E} \tag{A.15}$$

which differs from Eq. (A.11) since, as stated by Seweryn himself, the displacement field is not exact but approximated. Therefore the correct value of the discrete strain energy release is given by Eq. (A.11), whereas Eq. (A.15) contains an error. As a consequence, using Eq. (A.15), if one wants to recover, for crack propagation, the result $\sigma = \sigma_u$ for $a = 0$, it must be:

$$\varDelta = 0.474\left(\frac{K_{Ic}}{\sigma_u}\right)^2 \tag{A.16}$$

that, since affected by an error, differs from Eq. (A.13). Note that Eq. (A.16) is Eq. (A.19) of the paper by Seweryn (provided that he used the symbol l_0 instead of $\varDelta$).

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