

# Fractal fragmentation theory for size effects of quasi-brittle materials in compression

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*Size effects on dissipated energy density and strength for quasi-brittle materials in compression are analysed theoretically and experimentally herein. By using a fractal fragmentation approach the dissipated energy density of a structural element under compression is deduced. In addition, the dissipated energy density and the strength for geometrically similar structural elements under compression, by varying their size, are obtained. Finally, a comparison between theoretical predictions and experimental results is presented for plane as well as fibre-reinforced concrete. The size effects on dissipated energy density and compressive strength are captured by the proposed model in a satisfactory way.*

## Notation

$A_f$	total fracture surface area of fragments
$D$	fractal exponent
$N_p$	total number of fragments
$p$	probability size distribution function for fragments
$P$	cumulative size distribution function for fragments
$r$	fragment size
$r_{\min}$	size of the smallest fragment
$r_{\max}$	size of the largest fragment
$V = l^3$	specimen volume under compression
$V_f = l_f^3$	fragmented volume
$W$	energy dissipated during fragmentation
$\Psi = W/V$	dissipated energy density under compression
$\sigma_C$	material strength
$\beta$	friction exponent
$l_{ch}$	characteristic internal length

Symbols with '0' refer to a reference specimen.

## Introduction

Experimental results show that quasi-brittle materials, such as concrete, rocks or ceramics, demonstrate the dependence on specimen size of both energy density dissipation and nominal stress at the ultimate compressive load. Fracture mechanics has been used to study this phenomenon, which is called the size effect, as it cannot be explained by the classical continuum mechanics theories, such as elasticity or plasticity.

Although the effect of size on the mechanical properties of materials is very important, small-scale specimens are used to predict the behaviour of real structures and, although already analysed by a number of authors, the study of this phenomenon still does not provide a complete and systematic treatment. Such a phenomenon was investigated earlier by Griffith<sup>1</sup> and Weibull.<sup>2</sup> In the study by Griffith,<sup>1</sup> for the case of glass filaments the existence of inherent microcracks of a size that is proportional to the cross-sectional diameter of the filaments, is assumed as the basis of the size effect on nominal tensile strength. On the other hand, a purely statistical explanation of the same phenomenon using the weakest link concept can be given.<sup>2</sup>

Recently, the assumption of geometrical multifractality (i.e. scale-dependent fractal) for the damaged material microstructure represents the basis for the multifractal scaling law (MFSL) for tensile strength,<sup>3</sup> extrapolated for compressive strength in Carpinteri *et al.*<sup>4</sup>

The investigations carried out by Carpinteri *et al.*<sup>5,6</sup> emphasised the influence of friction and slenderness

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(MCR 31137) Paper received 12 May 2003; last revised 6 September 2004; accepted 1 December 2004

(shape effects) both numerically and experimentally as well as theoretically.<sup>7</sup>

The present investigation deals with this topic from a theoretical point of view based on fractal fragmentation models.<sup>7-10</sup> Several theoretical models have been proposed linking fractals<sup>11,12</sup> to fracture and fragmentation.<sup>13</sup> These models have been recently reviewed by Perfect.<sup>14</sup> However, the fragmentation theory has only been applied to the study of compression more recently.<sup>7,15</sup>

In particular, the paper extends the fractal explanation of the shape effects in compression, presented by Carpinteri and Pugno,<sup>15</sup> to the size effects and multi-fractal phenomena during compression of quasi-brittle materials. The theory will be compared with experiments under compression of both plane and reinforced concrete specimens.

### Energy dissipation under fragmentation

The aim of this section is the evaluation of the energy dissipation during fragmentation of a solid volume. We will assume the experimental self-similar distribution function for the size of the fragments and that the energy dissipated (due to crack formations and internal friction between the crack surfaces) is proportional to the total surface of the fragments. Since the result will be that the energy scale is not proportional to the fragmented volume, a size effect on the energy density will be obtained. Consequently a size effect on the strength of the solid will also be derived. Thus, we start by assuming after comminution or fragmentation, the cumulative distribution of particles  $P$  with radius smaller than  $r$  ( $r_{\min} \ll r_{\max}$ , with  $r_{\min}$  and  $r_{\max}$  being the radii of the smallest and of the largest fragment respectively) is<sup>7,8,16</sup>

$$P(< r) = 1 - \left(\frac{r_{\min}}{r}\right)^D \quad (1)$$

where  $D$  is a constant, the so-called fractal exponent, usually having a value between 2 and 3.

The probability density function  $p(r)$  times the interval amplitude  $dr$  represents the percentage of particles with radius values between  $r$  and  $r + dr$ . It is provided by derivation of the cumulative distribution function (equation (1))

$$p(r) = \frac{dP(< r)}{dr} = D \frac{r_{\min}^D}{r^{D+1}} \quad (2)$$

The total fracture surface area  $A_f$  is obtained by integration:

$$A_f = \int_{r_{\min}}^{r_{\max}} N_p (4\pi r^2) p(r) dr \cong 4\pi N_p \frac{D}{D-2} r_{\min}^2 \quad (3)$$

where  $N_p$  is the total number of particles.

On the other hand, the total volume of the particles (or fragmented volume  $V_f$ ) is:

$$V_f = \int_{r_{\min}}^{r_{\max}} N_p \left(\frac{4}{3}\pi r^3\right) p(r) dr \cong \frac{4}{3}\pi N_p \frac{D}{3-D} r_{\min}^D r_{\max}^{3-D} \quad (4)$$

If we assume a material ‘quantum’ of size  $r_{\min} = \text{constant}$ ,<sup>7,8,17,18</sup> and a hypothesis of self-similarity, namely,  $r_{\max} \propto \sqrt[3]{V_f}$  (the larger the fragmented volume, the larger the largest fragment,<sup>7,8,19</sup>) the energy  $W$ ,<sup>20</sup> dissipated in the comminution process into heat (by internal friction) and into surface energy (by fracture), is proportional to the total surface area  $A_f$ ,<sup>1,21</sup> and can be obtained by eliminating the particle number from equations (3) and (4)

$$W \propto A_f \propto V_f^{D/3} \quad (5)$$

and represents an extension of the postulate:<sup>22</sup>  $W \propto V_f^{2.5}$  (called in literature the ‘third comminution theory’).

The extreme cases contemplated by equation (5) are represented by  $D = 2$ , surface theory,<sup>23,24</sup> when the dissipation really occurs on a surface ( $W \propto V_f^{2/3}$ ), and by  $D = 3$ , volume theory,<sup>24,25</sup> when the dissipation occurs in a volume ( $W \propto V_f$ ). The experimental cases of comminution are usually intermediate, as well as the size distribution for concrete aggregates due to Füller.<sup>26</sup> On the other hand, concrete aggregates frequently are a product of natural fragmentation or artificial comminution. If the material to be fragmented is concrete, we have therefore a double reason to expect a fractal response.

### Fractal scaling laws

A specimen under compression will be fragmented following the cumulative particle size distribution (equation (1)) with  $D$  having a value around 2.<sup>7,8,15</sup>

The fragmented volume  $V_f = l_f^3$  and the volume of the specimen under compression  $V = l^3$  are not necessarily coincident, so that a power-law relation is assumed<sup>7</sup>

$$l_f \propto l^\beta \quad (6)$$

where  $\beta$  is an unknown exponent.

In the extreme cases, the fragmented volume is independent of the specimen volume so that the exponent  $\beta$  is equal to zero, or they are directly proportional so that  $\beta$  is equal to one.

The exponent  $\beta$  allows the modelling of the friction boundary conditions between loading platens and specimen in the study of the shape effects varying the specimen slenderness.<sup>7</sup> For the study of the size effects it can be assumed to be close to 1, supposing the fragmented volume proportional to the specimen volume.

This assumption, intuitive for frictionless tests, can be assumed also for friction ones. As a matter of fact, the frictional shearing stresses acting at the interface produce triaxially confined regions near the bases where a multitude of microcracks propagate.<sup>5-7</sup> In other words, the micro-cracked confined region, namely the fragmented volume, will be proportional to the specimen volume.

From equations (5) and (6) we can evaluate the relative dissipated strain energy density  $\Psi = W/V$  during the compression of the specimen as a function of its characteristic length  $l$

$$\frac{\Psi}{\Psi_0} = \left(\frac{l}{l_0}\right)^{\beta D - 3} \quad (7)$$

where  $\Psi_0, l_0$  are the strain energy density and the characteristic length related to a reference specimen.

Based on equation (7), the size effects on the compressive strength  $\sigma_C$  can be estimated assuming negligible size effect on the Young's modulus, namely,  $\Psi \propto \sigma_C^2$ , so that

$$\frac{\sigma_C}{\sigma_C^0} = \left(\frac{l}{l_0}\right)^{\frac{\beta D - 3}{2}} \quad (8)$$

where  $\sigma_C^0$  is the strength of the reference specimen.

Equations (7) and (8) represent the two fundamental size effects laws, based on the developed fractal fragmentation theory, where  $D$  is close to 2 and  $\beta$  can be considered a best-fit parameter, expected to be of the order of unity.

#### Geometrical multifractal extension

A geometrical multifractality or a scale-dependent fractality is described by an exponent  $D$  that is slightly variable with the size scale. Even if a unique clear trend can not be deduced for the fractal exponent, in general it is expected to range from 2 to 3 by increasing the size scale  $l$ .<sup>27</sup> This correspond to a dissipation over a surface at small scale and over a volume at large scale. The transition described by equation (7) in which  $D$  is changing from 2 to 3 with increasing  $l$  can be described by the following equation<sup>27</sup>

$$\Psi = \Psi_\infty \left(1 + \frac{l_{ch}}{l}\right) \quad (9)$$

where  $\Psi_\infty, l_{ch}$  are characteristic constants (the energy density for an infinite-sized element and a characteristic internal length connected to the microstructure), so that the size effects on compressive strength can be described by the following law

$$\sigma_C = \sigma_C^\infty \sqrt{1 + \frac{l_{ch}}{l}} \quad (10)$$

where  $\sigma_C^\infty$  is a characteristic constant (the strength of an infinite-sized element). Thus, equations (9) and (10)

represent the geometrical multifractal extension to our previously considered fractal approach. It is worth noting that equation (10) coincides with the well-known MFSL.<sup>4</sup>  $\Psi_\infty, l_{ch}, \sigma_C^\infty$  can be considered as best-fitting parameters.

### Comparison between theoretical and experimental results

In this section, a comparison is presented between the experimental results carried out by Campione and Mindless<sup>28,29</sup> and the theoretical predictions for the size

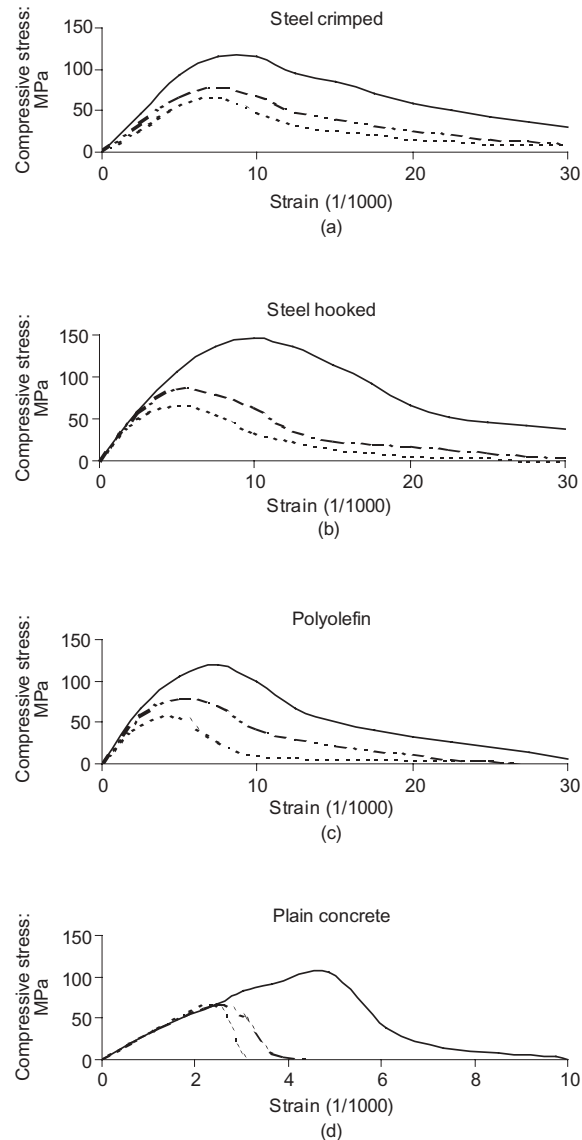


Fig. 1. Stress-strain experimental curves. Specimen of fibre-reinforced concrete (steel crimped (a), steel hooked (b), polyolefin (c)) or plane (d) concrete. Specimen sizes: 60 mm (solid line), 100 mm (solid/dashed line), 150 mm (dashed line).

effects on the dissipated energy density and on the compressive strength obtained respectively from the fractal laws of equations (7) and (8). The comparison regards geometrically similar cylindrical specimens with base diameters of 60, 100 and 150 mm and with fixed slenderness (height over diameter) equal to 2. The material is concrete with and without fibre reinfor-

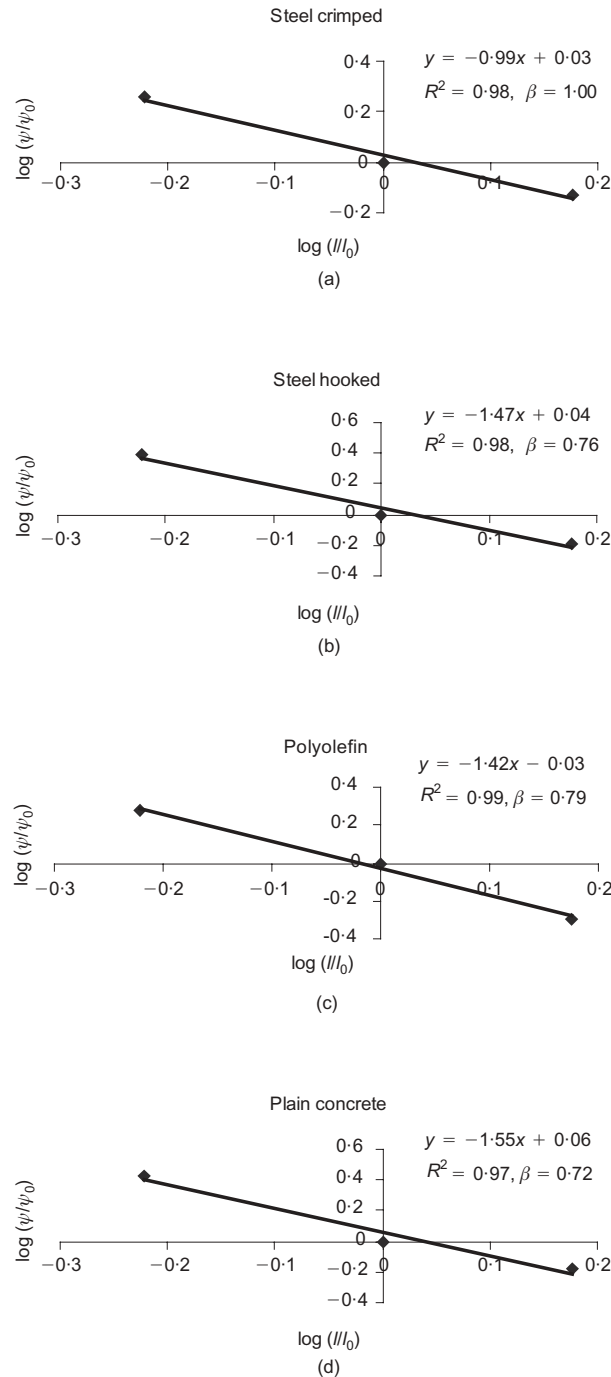


Fig. 2. Dissipated energy density against specimen size. Specimen of fibre-reinforced concrete (steel crimped (a), steel hooked (b), polyolefin (c) or plane (d) concrete. Comparison between theoretical straight line (equation (7) with best-fit parameter  $\beta$ ) and experimental points.

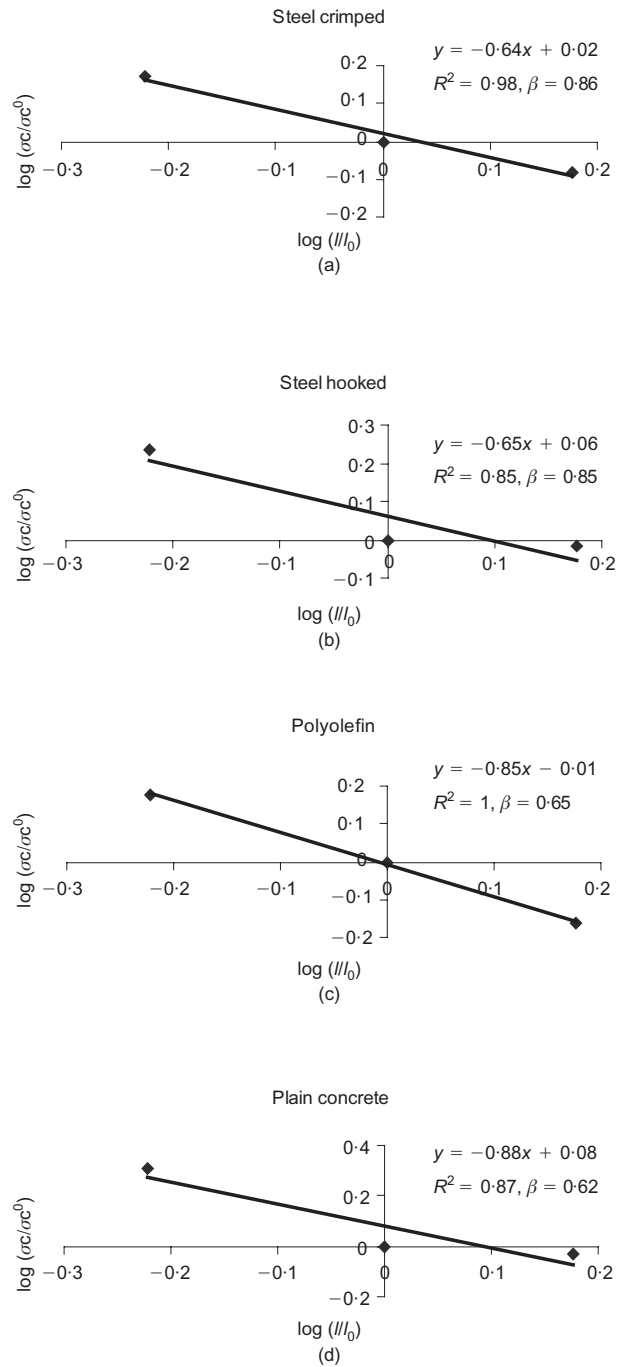


Fig. 3. Compressive strength against specimen size. Specimen of fibre-reinforced concrete (steel crimped (a), steel hooked (b), polyolefin (c) or plane (d) concrete. Comparison between theoretical straight line (equation (8) with best-fit parameter  $\beta$ ) and experimental points.

cement (plane concrete, steel crimped, steel hooked, polyolefin). The experimental stress-strain curves are reported in Fig. 1.

Equations (7) and (8), described by a straight line in a bi-logarithmic diagram, are experimentally confirmed (Figs 2 and 3). For each curve, the best-fit parameter  $\beta$  has been reported. As expected, the best-fit parameters  $\beta$  are of the order of the unity (slightly smaller than 1, in the analysed experiments; note that the imperfectly

linear trends observed in the bi-logarithmic diagrams show a positive concavity according to the geometrical multifractal laws).

## Conclusions

A very general law (equation (5)) describing the energy dissipation in natural or man-made fragmentation phenomena has been herein presented. It has been applied to the study of compression. As a consequence, the very simple scaling laws of equations (7) and (8), based on the developed fractal fragmentation theory, can be used to predict the size effects on energy density dissipation and strength for quasi-brittle materials in compression. Their geometrical multifractal extensions are equations (9) and (10).

The analysis of the results presented in the paper shows a correspondence between the theoretical predictions and the experimental data. The size effects are theoretically and quantitatively captured in a satisfactory way.

## Acknowledgements

The present research was carried out with the financial support of Ministry of University and Scientific Research (MURST), National Research Council (CNR) and EC-TMR Contract N° ERBFMRXCT960062.

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Discussion contributions on this paper should reach the editor by 1 February 2006