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Scale-effects on mean and standard deviation of the mechanical properties of condensed matter: an energy-based unified approach

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Abstract. The size effects on the mean values of the mechanical properties of condensed matter and on the related variances are analysed by means of a unified approach based on the multiscale character of energy dissipation. In particular, the scaling law for fragmentation energy density is obtained taking into account the self-similarity of fragments. It is based on a generalization of the three classical comminution laws that has been performed to evaluate the energy dissipation, computing volume and surface area of the particles for one- two- and three-dimensional fragmented objects. The result is general and can be applied to different fractal energy dissipation mechanisms, e.g., plasticity. Based on this approach, the scaling laws for mean and standard deviation values of the main mechanical properties of materials can be derived, like Young's and shear elastic moduli, ultimate normal and shear stresses and strains, fracture energy and toughness.

1. Introduction

Self-similar objects at all scales are actually well known in Nature (Mandelbrot 1982; Feder 1988); hence the use – and sometimes the misuse – of fractals can be found in the most diversified fields: energies, sizes and durations of solar flares, magnitudes of earthquakes, sizes of lakes, sizes of impact craters on moons, frequencies of usage of words, fragments of coal, size of asteroids, particles in the rings of Saturn, energy dissipation of warm-blooded animals, distribution of scales of coastlines, etc. For instance, the problem concerning the measurement of the length of Great Britain's coastline (Mandelbrot, 1967) showed the fractal nature of the coastline indentation. As a matter of fact, the Euclidean method for the measurement adopted at the beginning led to conflicting results, since the more accurate the measurement was, the more the coastline diverged. Likewise, objects with different dimensions, even belonging to various size-scales (e.g., a microfracture caused by fatigue in a metal and a macrofracture on the earth's crust originating from an earthquake) seem to be topologically similar. In other words, the morphology of a fracture appears self-similar at all scales: paradoxically, this does not allow researchers to understand whether the photograph of a fracture was taken with a microscope or a satellite. Therefore fractality is linked to *Scale Relativity*, making it impossible to establish an absolute scale.

With a view to pursuing such analysis in the field of fracture, a further consideration of fundamental interest is that at small scales the self-similarity phenomenon must fade away owing to quantization. Namely, it is to be expected that the Continuum should die down and be replaced by *Fracture Quanta* (Novozhilov, 1969). As far as a fracture is concerned, even if such fracture is in an extreme condition (e.g. crushing processes), it is therefore impossible to produce matter particles below a certain dimensional threshold (Material Quantum) because of a drastic energy increase to be spent in the process (Kendal, 1978). Hence, a serious technological difficulty derives in carrying out fine comminution, which is essential in a vast number of processes (e.g. sintering, medicine production...).

By combining both concepts, i.e., *Scale Relativity* or fractal geometry on the one hand, and the hypothesis of the existence of a *Fracture Quantum* on the other, this paper intends to derive the scaling laws for the energy density dissipation and for other mechanical properties of (quasi-brittle) materials, such as toughness, fracture energy, Young's and shear elastic moduli, and ultimate normal and shear stresses and strains.

2. Energy dissipations during fragmentation

Fragmentation involves particles at each scale. We assume a fractal (self-similar) particle size distribution, satisfying the Maximum Entropy Principle (Engleman et al., 1988) for the distribution in size of fragments (Carpinteri and Pugno, 2002a):

$$P(< r) = \frac{N(< r)}{N_0} = 1 - \left(\frac{a}{r}\right)^D \tag{1}$$

where $N(< r)$ is the number of fragments with size smaller than r, N_0 is the total number of fragments, $a(\ll r_{\max})$ is the minimum fragment size (or material quantum), and D is the so-called fractal exponent. Such a fractal exponent is theoretically positive. It is possible to observe experimentally that, in the vast majority of cases involving the crushing of three dimensional objects, such an exponent is comprised between 2 and 3 (e.g. disaggregated gneiss $D = 2.13$, disaggregated granite $D = 2.22$, broken coal $D = 2.50$, projectile fragmentation of quartzite $D = 2.55$, projectile fragmentation of basalt $D = 2.56$, fault gauge $D = 2.60$, sandy clays $D = 2.61$, terrace sands and gravels $D = 2.82$, glacial till $D = 2.88$) – see Turcotte (1992). This is theoretically equivalent to a crushing in which the smallest fragments provide the main contribution to the creation of the fracture surface, while the largest ones contribute to defining their volume. As a consequence, the fractal exponent seems to be close to a universal value, according to the universality observed by Bouchaud et al. (1990) for fracture surfaces. Fractal exponents outside this interval can be detected in a few cases, such as artificially crushed quartz ($D = 1.89$) or ash and pumice ($D = 3.54$). Usually for 2D-crushing, in which the smallest fragments provide the main contribution to the creation of the fracture perimeter while the largest ones define the area, the two-dimensional fractal exponent is comprised between 1 and 2. This is experimentally substantiated: in fact, several texts regarding ice floe fragmentation set the values of D as 1.7–1.8, 1.36 and 1.56 (Weiss, 2001). Likewise, a value ranging from 0 to 1 can be expected for the one-dimensional fractal exponent.

The probability density function $p(r)$ times the interval amplitude dr represents the percentage of particles with size between r and $r + \mathrm{d}r$. It is provided by derivation of the cumulative distribution function (1):

$$p(r) = \frac{\mathrm{d}P(< r)}{\mathrm{d}r} = D\frac{a^D}{r^{D+1}} \tag{2}$$

During fragmentation, the energy dissipation due to fracture, $\mathrm{d}W_F$, is proportional to the surface area of fragments, $\mathrm{d}A$ (Griffith, 1921):

$$\mathrm{d}W_F \propto \mathrm{d}A \tag{3}$$

During impact fragmentation (material in compression) the main energy dissipation $\mathrm{d}W_C$ is due to collisions and friction between particles (convertion into heat), and the effect is proportional to the same quantity $\mathrm{d}A$ (Smekal, 1937):

$$\mathrm{d}W_C \propto \mathrm{d}A \tag{4}$$

On the other hand, during explosion fragmentation (material in tension) the main energy loss $\mathrm{d}W_T$ is proportional to the kinetic energy of fragmented ejecta $\mathrm{d}K$. The velocity of fragmented ejecta is inversely proportional to the square root of fragment size as $\nu \propto r^{-1/2}$ (Nakamura and Fujiwara, 1991), so that the energy loss (kinetic energy) is proportional also in this case to the fragment surface $\mathrm{d}A$:

$$\mathrm{d}W_T \propto \mathrm{d}K \propto \nu^2 \mathrm{d}V \propto \mathrm{d}A \tag{5}$$

The energy spent into wave propagation, contributing to the same fragmentation process, will be a portion of the kinetic energy. An interesting and detailed analysis of this effect has been recently proposed by Simonov (2002).

Summarising, the global energy dissipation in impacts ($\mathrm{d}W_C + \mathrm{d}W_F$) or explosions ($\mathrm{d}W_T + \mathrm{d}W_F$), as well as in a mixed fragmentation ($\mathrm{d}W_C + \mathrm{d}W_T + \mathrm{d}W_F$), surprisingly appears to be proportional to the surface area $\mathrm{d}A$ of fragments.

3. Scaling law for fragmentation energy density: mean value

Based on Fracture Mechanics, we can make a statistical hypothesis of self-similarity, i.e., $r_{\max} \propto R$, with R the characteristic size of the object (the larger the fragmented object, the larger the largest fragment, see Carpinteri and Pugno, 2002a). Let us consider one- ($\mathsf{D} = 1$), two- ($\mathsf{D} = 2$) or three-dimensional ($\mathsf{D} = 3$) self-similar objects (Figure 1).

The total surface area of fragments can be obtained by integration ($\int_{r_{\min}}^{r_{\max}} = \int_a^{\propto R} \propto \int_a^R$):

$$A \propto \int_a^R N_0 r^{\mathsf{D}-1} p(r)\mathrm{d}r \cong \begin{cases} N_0 \dfrac{D}{D - \mathsf{D} + 1} a^{\mathsf{D}-1}, & D > \mathsf{D} - 1 \\ N_0 \dfrac{D}{\mathsf{D} - D - 1} a^D R^{\mathsf{D}-D-1}, & D < \mathsf{D} - 1. \end{cases} \tag{6}$$

On the other hand, the total volume of the particles, or total fragmented volume V, is:

$$V \propto \int_a^R N_0 r^{\mathsf{D}} p(r)\mathrm{d}r \cong \begin{cases} N_0 \dfrac{D}{\mathsf{D} - D} a^D R^{\mathsf{D}-D}, & D < \mathsf{D} \\ N_0 \dfrac{D}{D - \mathsf{D}} a^{\mathsf{D}}, & D > \mathsf{D}. \end{cases} \tag{7}$$

The energy per unit volume W_C dissipated during fragmentation, which is proportional to the total surface area A over the total fragmented volume V, can be obtained by eliminating N_0 from Equations (6) and (7):

$$W_C \propto A/R^{\mathsf{D}} \propto R^{-\frac{1}{2}+\delta}, \quad \text{with} \quad \begin{cases} \delta = -1/2, & D < \mathsf{D} - 1 \\ \delta \equiv D - \mathsf{D} + 1/2, & \mathsf{D} - 1 \leqslant D \leqslant \mathsf{D} \\ \delta = +1/2, & D > \mathsf{D}. \end{cases} \tag{8}$$

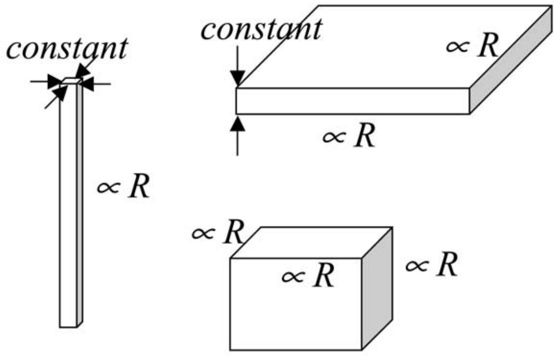

Figure 1. One-, two- and three-dimensional objects.

It is important to emphasize that $W_C \propto R^{\alpha}$, $-1 \leq \alpha \leq 0$, independently of the topological dimension D. In Equation (8), we have extended the result also to the case of equalities, for which only logarithmic corrections would appear.

The three-dimensional law for fragmentation in Equation (8) represents an extension of the Third Comminution Theory (Bond, 1952) obtained for $\delta = 0$, where *energy* $\propto V^{2.5/3}$, as well as its limit cases ($\delta = -1/2$ and $\delta = 1/2$), coincide respectively with the Surface Theory (von Rittinger, 1867), when the dissipation really occurs on a surface (*energy* $\propto V^{2/3}$), and with the Volume Theory (Kick, 1885), when the dissipation occurs in a volume (*energy* $\propto V$) – see Béla Beke (1964). Equation (8) shows, in its three-dimensional form, that the dissipation occurs in a fractal domain always comprised between surface and volume. It has successfully been applied by Carpinteri and Pugno (2002b,c and 2003) in different scientific areas.

The result of Equation (8) can be considered of wider validity and not only for fragmentation processes. For example, considering a bar under traction with self-similar distributed plastic zones, the scaling of the energy dissipation, which is proportional to the total volume of the plastic zones, can be obtained from Equation (8), in which $\mathsf{D} = 1$. In general, fractal dimensions of dislocation cell structures (i.e., where the energy is dissipated) have been observed comprised between 2 and 3 (in [100]-oriented Cu single crystals; Hahner, Bay, Zaiser, 1998), as here discussed for the case of $\mathsf{D} = 3$.

It is important to note that Equation (8) involves only the average value of the dissipation energy density.

4. Scaling law for fragmentation energy density: standard deviation

Considering as previously done $a = \text{const} \ll r_{\max} \propto R$, the scaling law for the standard deviation of the fragmentation energy density can be obtained as:

$$\sigma_W \propto \frac{1}{V}\sqrt{\int_a^R N_0(r^{\mathsf{D}-1} - \langle r^{\mathsf{D}-1}\rangle)^2 p(r)\mathrm{d}r} \tag{9}$$

where $\langle r^{\mathsf{D}-1}\rangle \propto A/N_0$ is the mean value of $r^{\mathsf{D}-1}$, substantially the mean fragment surface area. Introducing the expression for N_0 obtained from Equation (7) into Equation (9), as well as $a = \text{const} \ll r_{\max} \propto R$, we obtain the scaling for the standard deviation of the energy density:

$$\sigma_W \propto R^{-1-\frac{\varepsilon}{2}}, \quad \text{with} \quad \begin{cases} \varepsilon = 0, & D < \mathsf{D} \\ \varepsilon \equiv D - \mathsf{D}, & \mathsf{D} \le D \le \mathsf{D} + 1 \\ \varepsilon = 1, & D > \mathsf{D} + 1. \end{cases} \tag{10}$$

It is important to emphasize that $\sigma_W \propto R^{\alpha}$, $-\frac{3}{2} \le \alpha \le -1$, independently of the topological dimension D. Usually $D < \mathsf{D}$, so that $\varepsilon = 0$ and $\sigma_W \propto R^{-1}$. *Larger dispersions are predicted for smaller structures*.

Summarizing, the scaling for the average value of the energy density dissipated during fragmentation is given by Equation (8), as well as the scaling for the related standard deviation is given by Equation (10).

5. Scaling laws for elastic moduli: mean values

Let us assume a block of condensed matter containing several self-similar pre-existing defects like cracks and pores (a three-dimensional deterministic idealization could be represented by the well-known Menger sponge). According to the fractal hypothesis, the effective volume of the block scales as $R^{\mathsf{D}-\eta}$, where $0 \le \eta \le 1$ is connected to the fractal dimension of defects (e.g., pores in the Menger sponge); a solid without pre-existing defects is described by $\eta = 0$, as well as $\eta = 1$ describes the opposite limit case (maximum distribution of defects). As a consequence, the volume fraction of matter (i.e., the effective volume of matter over the total) scales as $R^{-\eta}$, R^{D} being proportional to the nominal volume of the solid. A classical rule of mixture for the Young modulus E gives its scaling:

$$E = E_m \nu_m + E_d \nu_d \propto R^{-\eta} \tag{11}$$

where the intrinsic Young modulus E_m of the matter is a constant, the Young modulus E_d of defects is equal to zero by definition and $\nu_m = 1 - \nu_d \propto R^{-\eta}$ is the previously mentioned volume fraction. The same conclusion could be obtained for the shear elastic modulus G. As a consequence, the Poisson's ratio is predicted to be size-independent. Summarizing:

$$E \propto G \propto R^{-\eta}. \tag{12}$$

It is important to emphasize that $E \propto G \propto R^{\alpha}$, $-1 \le \alpha \le 0$, independently of the topological dimension D.

The result of this very simple model agrees with the experimental evidence that smaller is stiffer (Treacy, Ebbesen and Gibson, 1996).

6. Scaling laws for the other mechanical properties: mean values

By applying the scaling law for the mean energy density of Equation (8), the scaling laws for the average mechanical properties of condensed matter, like critical normal and shear stresses σ_C, τ_C and strains ε_C, γ_C, fracture energy G_C and toughness K_{IC}, can be easily obtained by expressing the energy density in the following different forms:

$$W_C \propto \frac{\sigma_C^2}{E} \propto \frac{\tau_C^2}{G} \propto E\varepsilon_C^2 \propto G\gamma_C^2 \propto \frac{\mathsf{G}_C}{R} \propto \frac{K_{IC}^2}{ER}. \tag{13}$$

It is important to emphasize that Equation (13) should be verified also for nonlinear behaviours, the constants of proportionality for stresses and strains being in general different from $^1/_2$. From Equations (8) and (12), by means of Equation (13), the scaling laws for the mean mechanical properties can be obtained as:

$$\mathsf{G}_C \propto R^{\frac{1}{2}+\delta} \propto R^{\alpha}, \qquad 0 \le \alpha \le 1 \tag{14}$$

$$K_{IC} \propto R^{\frac{1}{4}+\frac{\delta}{2}-\frac{\eta}{2}} \propto R^{\alpha}, \qquad -\frac{1}{2} \le \alpha \le \frac{1}{2} \tag{15}$$

$$\sigma_C \propto \tau_C \propto R^{-\frac{1}{4}+\frac{\delta}{2}-\frac{\eta}{2}} \propto R^{\alpha}, \qquad -1 \le \alpha \le 0 \tag{16}$$

$$\varepsilon_C \propto \gamma_C \propto R^{-\frac{1}{4}+\frac{\delta}{2}+\frac{\eta}{2}} \propto R^{\alpha}, \qquad -\frac{1}{2} \le \alpha \le \frac{1}{2} \tag{17}$$

The results show us that the fracture energy increases by increasing the size R, as well as the strengths increase by decreasing the size: 'smaller is stronger' is a well-known experimental evidence. The classical size effect on strength predicted by Fracture Mechanics corresponds to the intermediate case of $\sigma_C \propto R^{-\frac{1}{2}}$. The lowest power for strength is predicted to be -1. On the other hand, both critical strain and fracture toughness could theoretically increase by increasing or decreasing the structural size. In addition, these inversions of tendencies, for fracture energy (as well as for the strength) are not allowed.

Furthermore, the Fractal Scaling Laws for σ_C, ε_C and G_C, introduced by Carpinteri (Carpinteri 1994a,b and Carpinteri et al. 2002) in a different way (based on the order to disorder transition), are:

$$\begin{aligned} &\text{Fractal Scaling Laws} : \mathsf{G}_C \propto R^{d_{\mathsf{G}}}, \sigma_C \propto R^{-d_\sigma}, \varepsilon_C \propto R^{-d_\varepsilon}, \\ &\text{with} \quad d_{\mathsf{G}} + d_\sigma + d_\varepsilon = 1. \end{aligned} \tag{18}$$

A comparison between Equations (14), (16), (17) and (18) shows that the relation $d_{\mathsf{G}}+d_\sigma+d_\varepsilon = 1$ is identically satisfied for each couple (δ, η). For these reasons, Equations (14), (16) and (17) can be considered a generalization of the (mono) fractal scaling laws, Equation (18).

7. Scaling laws for the other mechanical properties: standard deviations

The scaling laws for the standard deviations of the critical mechanical properties can be estimated for negligible elastic moduli dispersions:

$$\sigma_E = \sigma_G = 0. \tag{19}$$

By derivation of Equation (13), we obtain the corresponding relationship on standard deviations:

$$\sigma_W \propto \frac{\sigma_C \sigma_\sigma}{E} \propto \frac{\tau_C \sigma_\tau}{G} \propto E\varepsilon_C\sigma_\varepsilon \propto G\gamma_C\sigma_\gamma \propto \frac{\sigma_{\mathsf{G}}}{R} \propto \frac{K_{IC}\sigma_K}{ER}. \tag{20}$$

From the scaling laws of Equations (10) and (19), by means of Equation (20), the scaling laws for the standard deviations of the mechanical properties can be obtained as:

$$\sigma_\sigma \propto \sigma_\tau \propto R^{-\frac{3+2(\delta+\varepsilon+\eta)}{4}} \propto R^{\alpha}, \qquad -2 \le \alpha \le -\frac{1}{2} \tag{21}$$

Table 1. Scaling Laws ($-1/2 \le \delta \le 1/2, 0 \le \eta, \varepsilon \le 1$).

Property	Mean Value	Standard Deviation
Critical energy density	$W_C \propto R^{-\frac{1}{2}+\delta}$	$\sigma_W \propto R^{-1-\frac{\varepsilon}{2}}$
Fracture energy	$\mathsf{G}_C \propto R^{\frac{1}{2}+\delta}$	$\sigma_{\mathsf{G}} \propto R^{-\frac{\varepsilon}{2}}$
Fracture toughness	$K_{IC} \propto R^{\frac{1}{4}+\frac{\delta}{2}-\frac{\eta}{2}}$	$\sigma_K \propto R^{-\frac{1}{2}(\frac{1}{2}+\delta+\varepsilon+\eta)}$
Critical stresses	$\sigma_C \propto \tau_C \propto R^{-\frac{1}{4}+\frac{\delta}{2}-\frac{\eta}{2}}$	$\sigma_\sigma \propto \sigma_\tau \propto R^{-\frac{3+2(\delta+\varepsilon+\eta)}{4}}$
Critical strains	$\varepsilon_C \propto \gamma_C \propto R^{-\frac{1}{4}+\frac{\delta}{2}+\frac{\eta}{2}}$	$\sigma_\varepsilon \propto \sigma_\gamma \propto R^{-\frac{3+2(\delta+\varepsilon-\eta)}{4}}$
Elastic moduli	$R \propto G \propto R^{-\eta}$	$\sigma_E = \sigma_G = 0$

Table 2. Scaling Laws (extreme cases).

Scaling $\propto R^\alpha$	Mean Value	Standard Deviation
Dissipated energy density	$-1 \le \alpha \le 0$	$-\frac{3}{2} \le \alpha \le -1$
Fracture energy	$0 \le \alpha \le 1$	$-\frac{1}{2} \le \alpha \le 0$
Fracture toughness	$-\frac{1}{2} \le \alpha \le \frac{1}{2}$	$-\frac{3}{2} \le \alpha \le 0$
Critical stresses	$-1 \le \alpha \le 0$	$-2 \le \alpha \le -\frac{1}{2}$
Critical strains	$-\frac{1}{2} \le \alpha \le \frac{1}{2}$	$-\frac{3}{2} \le \alpha \le 0$
Elastic moduli	$-1 \le \alpha \le 0$	$\sigma_E = \sigma_G = 0$

$$\sigma_\varepsilon \propto \sigma_\gamma \propto R^{-\frac{3+2(\delta+\varepsilon-\eta)}{4}} \propto R^\alpha, \qquad -\frac{3}{2} \le \alpha \le 0 \tag{22}$$

$$\sigma_{\mathsf{G}} \propto R^{-\frac{\varepsilon}{2}} \propto R^\alpha, \qquad -\frac{1}{2} \le \alpha \le 0 \tag{23}$$

$$\sigma_K \propto R^{-\frac{1}{2}(\frac{1}{2}+\delta+\varepsilon+\eta)} \propto R^\alpha, \qquad -\frac{3}{2} \le \alpha \le 0. \tag{24}$$

It is important to emphasize that they are independent of the topological dimension D and that all the predictions are of larger dispersions for smaller structures. Obviously, the predicted scaling of the standard deviations are based on the hypothesis of Equation (19).

8. Conclusions

Assuming a fractal statistical distribution for the energy dissipated into condensed matter, the scaling laws for the mean values and for the standard deviations of the related mechanical properties have been estimated. The scaling laws are summarized in Table 1. In Table 2 their extreme cases are emphasized. Reference values for the power-law exponents can be considered assuming an energy dissipation arising over a fractal domain exactly intermediate between a surface and a volume ($\delta = 0$), a material with negligible Young modulus

Table 3. Scaling Laws (reference values: $\delta = \eta = \varepsilon = 0$)

Scaling $\propto R^{\alpha}$	Mean Value	Standard Deviation
Dissipated energy density	$\alpha = -\frac{1}{2}$	$\alpha = -1$
Fracture energy	$\alpha = \frac{1}{2}$	$\alpha = 0$
Fracture toughness	$\alpha = \frac{1}{4}$	$\alpha = -\frac{1}{4}$
Critical stresses	$\alpha = -\frac{1}{4}$	$\alpha = -\frac{3}{4}$
Critical strains	$\alpha = -\frac{1}{4}$	$\alpha = -\frac{3}{4}$
Elastic moduli	$\alpha = 0$	$\sigma_E = \sigma_G = 0$

scaling ($\eta = 0$), as well as the usual case of a fractal exponent lower than the corresponding topological dimension ($\varepsilon = 0$). The corresponding reference values are summarized in Table 3.

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