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Dynamic response of damped von Koch antennas

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Abstract
In this paper, the mechanical behavior of a von Koch beam, with different indentation angles, is investigated. Reductions of stiffness, mass and damping matrices lead to simple analytical recursive relationships depending on the fractal dimension of the structure. Results are then exploited to perform a complete modal analysis, which suggests peculiar scaling laws for the natural and damped frequencies. Eventually, the response of the structure to the unit step function is considered. The analysis suggests a new methodology to optimize the damping response of fractal antennas.

Keywords
Fractal antennas, generic von Koch beam, modal analysis

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1. Introduction

The von Koch beam is one of the most well-known fractal structures (von Koch, 1906), which turns out to be particularly suitable to design fractal antennas. Indeed, fractal-shaped antennas have some unique characteristics that are linked to the geometrical properties of fractals (Cohen, 1996; Puente et al., 2000). Firstly, because the self-similarity property, which means that the structure is composed by some parts having the same shape as the whole but at a different scale, makes fractals specially suitable to design multi-frequency antennas. Secondly, because the huge space-filling properties of some fractal shapes, described by the fractal dimension, help in the realization of small antennas to better take advantage of the surrounding space.

The properties of a von Koch curve have recently been widely investigated: the existence of a homeomorphism between the closed real interval [0,1] and the von Koch curve has been proved in (Epstein and Śniatycki, 2008), while an analysis on the surface contained inside a von Koch snowflake has been developed in (Milosŏević and Ristanovic, 2007). On the other hand, with regard to a von Koch beam, considered a hierarchical Eulero-Bernoulli framed-beam structure, its static mechanical behavior has been analyzed, both numerically (Epstein and Adeeb, 2008), by means of a self-similarity postulate, and analytically (Carpinteri et al., 2009), by means of recursive relationships on the strain energy and stiffness matrix. In the latter case, results have then been exploited to perform a complete free-vibration analysis of the structure (Carpinteri et al., 2010), with particular focus into the resonant frequencies. Thanks to matrix reduction (Guyan, 1965; Bouhaddi and Fillod, 1994; Lin and Xia, 2003), simple recursive scaling laws are provided.

In this paper, the mechanical behavior investigation is extended to a generic von Koch beam i.e., with a generic indentation angle, and to the damped case. The paper is structured as follows: in Section 2, the von Koch beam construction is briefly recalled as well as the equation which describes how the fractal dimension of the structure varies as the indentation angle varies. Stiffness, mass and damping matrix scaling laws related to such structures are presented

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in Section 3. In Section 4, a complete modal analysis is performed and the resonant and damped resonant frequencies of von Koch cantilever beams are evaluated. Eventually, the response of the structures to the unit step function is investigated by means of a Finite Element (FE) analysis.

2. Generic von Koch beam

The classical von Koch beam is generated starting from a line segment of length l_0 (called *the intiator*): at each step the middle third of each segment is removed and replaced by the other two sides of the equilateral triangle based on the removed segment. In this case, an indentation angle $\theta = 60^\circ$ is taken into account and the fractal dimension of the structure is $D = \ln 4/\ln 3$. Let us remember that the dimension of a fractal set, as observed in Section 1, provides an useful description of how much space a set fills. This property, together with that of self-similarity, strongly affects the antennas' performance.

The construction may be generalized to different values of θ (Figure 1), then considering a different fractal dimension D^*

$$D^* = -\frac{\ln 4}{\ln q}, \qquad (1)$$

where

$$q = \frac{1}{2(1+\cos\theta)}. \qquad (2)$$

According to equation (1), for a von Koch beam, D^* is hence a monotonic increasing function of the angle θ, $0 \leq \theta < 90^\circ$.

3. Stiffness, mass and damping matrices

The static analysis of a von Koch beam has been widely investigated in (Carpinteri et al., 2009). Since at each iteration n the number of nodes (and hence of the degrees-of-freedom, (DOF)) grows exponentially as 2^{2n+1}, the dimensions of the stiffness and mass matrices increase. In Carpinteri et al. (2010) it has been proved that, by reducing matrices with respect to the same set of nodes (henceforth called master, as the related DOF), particular scaling laws, depending on the fractal dimension, emerge after different iterations of the structure. Starting from the results on the strain energy, obtained by considering a classical von Koch cantilever beam subjected at the free end to three different loading conditions, the reduced stiffness matrix $\mathbf{K}_n$ of a generic von Koch beam can be written as:

$$\mathbf{K}_n = (4q)^{1-n}\frac{k}{l_0^3}\bar{\mathbf{K}}_n = \frac{4qk}{l_0^3}\left(\frac{l_n}{l_0}\right)^{D^*-1}\bar{\mathbf{K}}_n, \quad n > 1, \qquad (3)$$

where l_0 is the length of the *initiator* ($n = 0$), $l_n = q^n \cdot l_0$ is the length of each rectilinear beam constituting the structure at the n-th step, k is the beam rigidity, i.e. the product of the Young's modulus E of the material times the moment of inertia I of the cross-section with respect to the neutral axis, and $\bar{\mathbf{K}}_n$ is the dimensionless stiffness matrix, which converges to finite values after approximately six iterations.

The stiffness matrix $\mathbf{K}_n$ in equation (3) scales asymptotically as $(4q)^{-n}$. For n tending to infinity, the structural stiffness trivially tends to zero and the structure becomes infinitely compliant.

On the other hand, taking into account the real distribution of the masses over the beam, the following recursive relationship of the mass matrix $\mathbf{M}_n$ is obtained:

$$\mathbf{M}_n = (4q)^{n-1}\frac{ml_0}{420}\bar{\mathbf{M}}_n = \frac{ml_0}{1680q}\left(\frac{l_n}{l_0}\right)^{1-D^*}\bar{\mathbf{M}}_n, \quad n > 1 \qquad (4)$$

$m = \rho A$ being the mass per unit length, where ρ is the material density and A is the area of the cross-section, and $\bar{\mathbf{M}}_n$ the dimensionless mass matrix, which converges to finite values after a few iterations. Equation (4) represents the counterpart of equation (3): while each term of the stiffness matrix tends to vanish (by scaling asymptotically as $(4q)^{-n}$) as the number of iterations n increases, the coefficients of the mass matrix diverge (by scaling as $(4q)^n$ i.e., exactly as the total length $L_n = 4^n l_n = (4q)^n l_0$ of the structure).

The validity of equations (3) and (4) for a generic angle θ different from 60°, which has been implicitly assumed so far, will be proved in the next section by considering the related modal analysis.

Eventually, let us consider the special case in which the symmetric damping matrix $\mathbf{C}_n$ is a linear combination of the matrices $\mathbf{M}_n$ and $\mathbf{K}_n$, namely when:

$$\mathbf{C}_n = \alpha_n\mathbf{M}_n + \beta_n\mathbf{K}_n, \quad n > 1, \qquad (5)$$

where α_n and β_n are real constants. This damping model is also known as ''proportional damping'' or ''Rayleigh damping''. Modes of proportionally damped systems preserve the simplicity of the real normal modes as in the undamped case. In Section 4, suitable scaling laws for the two constants will be provided.

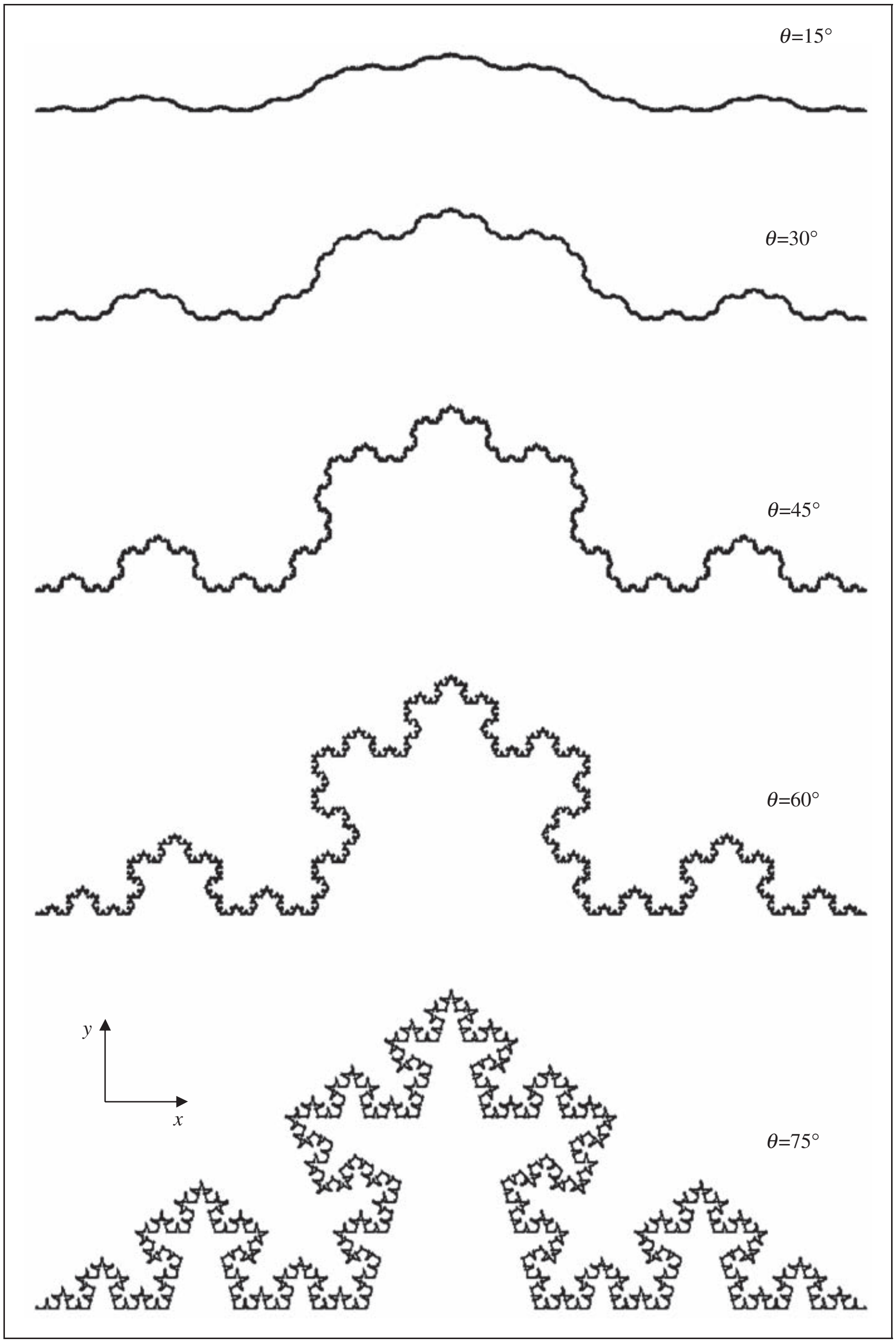

Figure 1. Generic von Koch beams at the iteration $n = 5$.

Observe that stiffness and mass matrices (and consequently the proportional damping one) remain finite as n increases only if the beam rigidity k and the mass per unit length m scale as $(4q)^n$ and $(4q)^{-n}$, respectively.

4. Modal analysis

In Carpinteri et al. (2010) it is proved, by means of Guyan's reduction, that the choice of reducing stiffness and mass matrices of a von Koch beam with respect to the six DOF of the two extreme nodes is sufficient for the investigation of the first two natural vibrating frequencies. Increasing the number N of master DOF the number of modes which can be accurately analyzed reasonably increases (Bouhaddi and Fillod, 1994). By considering as masters the 15 DOF related to the five nodes of the first order von Koch beam, for instance, evaluated frequencies are precise up to the seventh mode (Carpinteri et al., 2010). They become totally diverging from the real ones only above the tenth mode. This choice, how it will be shown in Section 4.2, will reveal sufficient to investigate accurately the damped response of von Koch antennas.

Eventually, note that self-similarity of the structure can be exploited to simplify calculations (Epstein and Adeeb, 2008).

4.1. Free vibration motion

Once the stiffness and mass matrices are known (equations (3) and (4)), the governing differential equation of motion of a von Koch beam, in its free natural vibration, can be written as:

$$\mathbf{M}_n\ddot{\boldsymbol{\delta}}_n + \mathbf{K}_n\boldsymbol{\delta}_n = \mathbf{0}, \tag{6}$$

$\boldsymbol{\delta}_n$ and $\ddot{\boldsymbol{\delta}}_n$ being the vectors of nodal displacements and of the corresponding accelerations, respectively, at the iteration n. In order to investigate the free oscillation of the system, let us suppose that the generalized coordinates vary harmonically in time t as:

$$\boldsymbol{\delta}_n = (\boldsymbol{\delta}_0)_n \sin\omega_n t, \tag{7}$$

where the angular frequencies ω_n and maximum amplitudes $(\boldsymbol{\delta}_0)_n$ are to be determined via the eigenvalue problem:

$$\left(\mathbf{K}_n - \omega_n^2\mathbf{M}_n\right)(\boldsymbol{\delta}_0)_n = \mathbf{0}. \tag{8}$$

By solving equation (8), the following natural frequency scaling law is obtained:

$$\omega_{i,n} = (4q)^{1-n} a_{i,n}^{(\omega)}\omega_{i,1} = 4q\left(\frac{l_n}{l_0}\right)^{D^*-1} a_{i,n}^{(\omega)}\omega_{i,1}, \quad i = 1, \ldots, N, \tag{9}$$

where the first subscript refers to the mode, while the second one refers to the von Koch beam iteration ($\omega_{1,n}$, for instance, is the fundamental frequency related to the n-th order iteration). The fundamental frequency behavior of a von Koch cantilever beam is reported in Figure 2, while the coefficients $a_{i,n}^{(\omega)}$ related to the first three natural frequencies are reported in Table 1: if four decimal digits are taken into account, convergence is expected after nearly six iterations.

Note that the $T_{i,n}$ period scaling law is trivially recovered by inverting equation (9). On the other hand, inserting it into equation (8) yields:

$$(\boldsymbol{\delta}_0)_{i,n} = \sqrt{4q}\left(\frac{l_n}{l_0}\right)^{\frac{D^*-1}{2}} \mathbf{a}_{i,n}^{(\delta)}(\boldsymbol{\delta}_0)_{i,1}, \quad i = 1, \ldots, N, \tag{10}$$

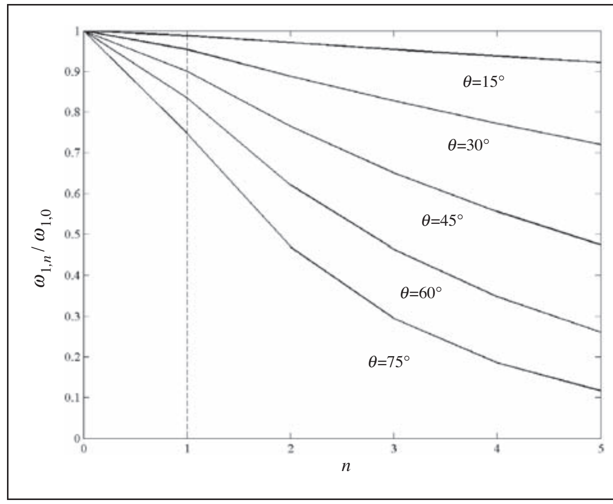

Figure 2. Dimensionless natural frequencies vs. iteration n, for different indentation angles θ.

Table 1. Coefficients $a_{i,n}^{(\omega)}$ related to the natural frequency scaling laws (Eq. (9))

Angle	Iteration n	1	2	3	4	5
15°	$a_{1,n}^{(\omega)}$	1.0000	0.9994	0.9990	0.9989	0.9989
	$a_{2,n}^{(\omega)}$	1.0000	1.0004	1.0008	1.0010	1.0011
	$a_{3,n}^{(\omega)}$	1.0000	0.9632	0.9618	0.9615	0.9614
30°	$a_{1,n}^{(\omega)}$	1.0000	0.9975	0.9964	0.9961	0.9961
	$a_{2,n}^{(\omega)}$	1.0000	1.0152	1.0182	1.0188	1.0193
	$a_{3,n}^{(\omega)}$	1.0000	0.9678	0.9636	0.9625	0.9622
45°	$a_{1,n}^{(\omega)}$	1.0000	0.9949	0.9926	0.9919	0.9918
	$a_{2,n}^{(\omega)}$	1.0000	1.0434	1.0515	1.0532	1.0536
	$a_{3,n}^{(\omega)}$	1.0000	0.9842	0.9811	0.9797	0.9793
60°	$a_{1,n}^{(\omega)}$	1.0000	0.9932	0.9899	0.9886	0.9884
	$a_{2,n}^{(\omega)}$	1.0000	1.0905	1.1084	1.1118	1.1125
	$a_{3,n}^{(\omega)}$	1.0000	1.0226	1.0251	1.0233	1.0225
75°	$a_{1,n}^{(\omega)}$	1.0000	0.9979	0.9934	0.9920	0.9917
	$a_{2,n}^{(\omega)}$	1.0000	1.1667	1.2047	1.2132	1.2152
	$a_{3,n}^{(\omega)}$	1.0000	1.0922	1.1138	1.1157	1.1157

the modes having been opportunely normalized with respect to the mass. $\mathbf{a}_{i,n}^{(\delta)}$ is the $N \times N$ diagonal matrix of the normalized eigenvector coefficients.

The physical soundness of the scaling laws provided by equations (9) and (10) is supported by introducing the Rayleigh's quotient:

$$\omega_{i,n}^2 = \frac{(\boldsymbol{\delta}_0)_{i,n}^T\mathbf{K}_n(\boldsymbol{\delta}_0)_{i,n}}{(\boldsymbol{\delta}_0)_{i,n}^T\mathbf{M}_n(\boldsymbol{\delta}_0)_{i,n}}, \quad i = 1, \ldots, N, \tag{11}$$

which consistently scales as $(4q)^{-2n}$.

4.2. Forced damped motion

The response of a general viscously damped system represents a considerably more difficult problem, due to the coupling introduced by damping. Also in this case, the differential equations of motion may be written in the matrix form:

$$\mathbf{M}_n\ddot{\boldsymbol{\delta}}_n + \mathbf{C}_n\dot{\boldsymbol{\delta}}_n + \mathbf{K}_n\boldsymbol{\delta}_n = \mathbf{F}_n, \tag{12}$$

where $\dot{\boldsymbol{\delta}}_n$ and $\mathbf{F}_n$ are the vectors of the nodal velocities and of the applied forces, respectively.

Let us now introduce the modal matrix $\boldsymbol{\Delta}_n$ (i.e., the matrix whose columns are the normalized eigenvectors provided by equation (10)) and the transformation

$$\boldsymbol{\delta}_n = \boldsymbol{\Delta}_n\boldsymbol{\eta}_n, \tag{13}$$

$\boldsymbol{\eta}_n = (\boldsymbol{\eta}(\mathrm{t}))_n$ being the normal coordinates. Inserting equation (13) into equation (12) yields

$$\ddot{\boldsymbol{\eta}}_n + \mathbf{c}_n\dot{\boldsymbol{\eta}}_n + \boldsymbol{\omega}_n^2\boldsymbol{\eta}_n = \mathbf{Q}_n, \tag{14}$$

where $\boldsymbol{\omega}_n^2$ is the $N \times N$ diagonal matrix of the natural angular frequencies, $\mathbf{Q}_n$ is the N modal force vector and $\mathbf{c}_n$ is a $N \times N$ symmetric matrix, generally non-diagonal. In the proportional damping case (equation (5)), $\mathbf{c}_n$ does indeed become diagonal (let us remember that equation (5) is a sufficient but not necessary condition to get $\mathbf{c}_n$ diagonal, (Caughey and O'Kelly, 1965):

$$\mathbf{c}_n = \alpha_n\mathbf{I} + \beta_n\boldsymbol{\omega}_n^2 = \begin{bmatrix} 2\zeta_{1,n}\omega_{1,n} & 0 & \dots & 0 \\ 0 & 2\zeta_{2,n}\omega_{2,n} & \dots & \dots \\ \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & 2\zeta_{N,n}\omega_{N,n} \end{bmatrix}, \tag{15}$$

where $\mathbf{I}$ is the $N \times N$ diagonal unit matrix and $\zeta_{i,n}$ is the modal damping factor

$$\zeta_{i,n} = \frac{1}{2}\left(\frac{\alpha_n}{\omega_{i,n}} + \beta_n\omega_{i,n}\right), \tag{16}$$

so that equation (14) reduces to an independent set of equations:

$$\ddot{\eta}_{i,n} + 2\zeta_{i,n}\omega_{i,n}\dot{\eta}_{i,n} + \omega_{i,n}^2\eta_{i,n} = Q_{i,n}, \quad i = 1, \ \dots, \ N. \tag{17}$$

It follows from equations (9) and (16) that, if α_n and β_n do not vary, $\zeta_{i,n}$ scales asymptotically as $(4q)^{n-1}$: in such a case, the von Koch beam tends to become an over-damped system for each mode. In the following, this result will not be further investigated both because flexible and oscillating structural behaviors are generally expected when dealing with antennas, and because it would be difficult to provide meaningful values for α_n and β_n at the start of the analysis. Indeed, the problem of computation of Rayleigh damping coefficients has been faced by several authors (see, for a deeper analysis, (Adhikari, 2006; Chowdhury and Dasgupta, 2008). The easiest practice consists in assuming a constant damping ratio ζ for all significant modes. On the other hand, since it is generally observed that ζ_i increases with increasing the mode order, it is not difficult to describe the Rayleigh damping by choosing $\zeta_i = \zeta$ (n fixed) for two modes in equation (16) and solving the corresponding damping coefficients α_n and β_n. Considering the first two frequencies, yields:

$$\alpha_n = \frac{2\zeta\omega_{1,n}\omega_{2,n}}{\omega_{1,n} + \omega_{2,n}} = (4q)^{1-n}\frac{2\zeta a_{1,n}^{(\omega)}a_{2,n}^{(\omega)}\omega_{1,1}\omega_{2,1}}{a_{1,n}^{(\omega)}\omega_{1,1} + a_{2,n}^{(\omega)}\omega_{2,1}}, \tag{18a}$$

$$\beta_n = \frac{2\zeta}{\omega_{1,n} + \omega_{2,n}} = (4q)^{n-1}\frac{2\zeta}{a_{1,n}^{(\omega)}\omega_{1,1} + a_{2,n}^{(\omega)}\omega_{2,1}}. \tag{18b}$$

Once ζ_i is evaluated, the damped frequency related the i-th mode clearly writes as:

$$(\omega_\mathrm{d})_{i,n} = \omega_{i,n}\left(1 - \zeta_i^2\right)^{1/2} = (4q)^{1-n}a_{i,n}^{(\omega)}\omega_{i,1}\left(1 - \zeta_i^2\right)^{1/2}. \tag{19}$$

Let us now investigate the damped response of a generic von Koch cantilever beam to a transversal unit step function $F = u(t)$, where $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t > 0$, applied at the free end (Figure 3). Null initial conditions are assumed. By means of equations (18a,b), values of α_n and β_n for all significant modes are evaluated, starting from a damping coefficient ζ equal to 0.05. LUSAS® code is used to perform FE simulations: the von Koch beams are considered as Euler-Bernoulli framed-beam structures and the 15 DOF related to the five nodes of the first order von Koch beam are chosen as masters (see Carpinteri et al., 2010). This choice is sufficient to include in the analysis all modes until the total sum of mass participation factors (MPFs) is greater than 85%. Each rectilinear beam of length l_n, constituting the structures at the iteration n, is properly meshed until numerical convergence is achieved: while for $n = 1$, 2 and 3 (independently of θ) the size of each element is taken equal to $l_n/12$, $l_n/3$ and $l_n/2$, respectively, for higher-order iterations ($n \geq 4$) the size of each element is assumed exactly equal to l_n. It is important to point out that results presented below do not differ significantly from those obtained without any reduction procedure, thus confirming the validity of the present approach.

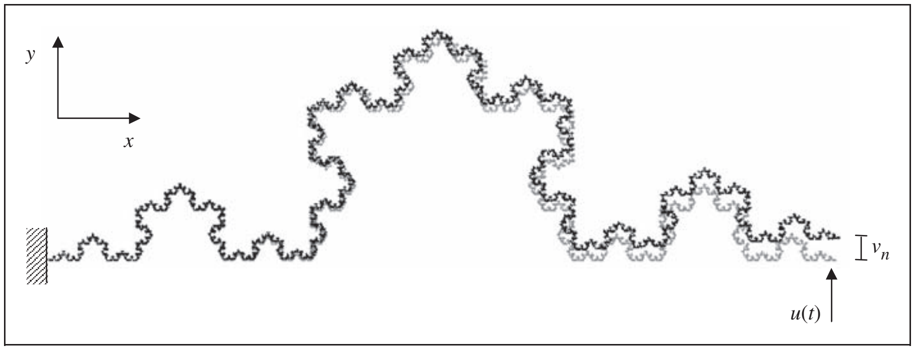

Figure 3. von Koch cantilever beam subjected to a unit step transversal force at the free end.

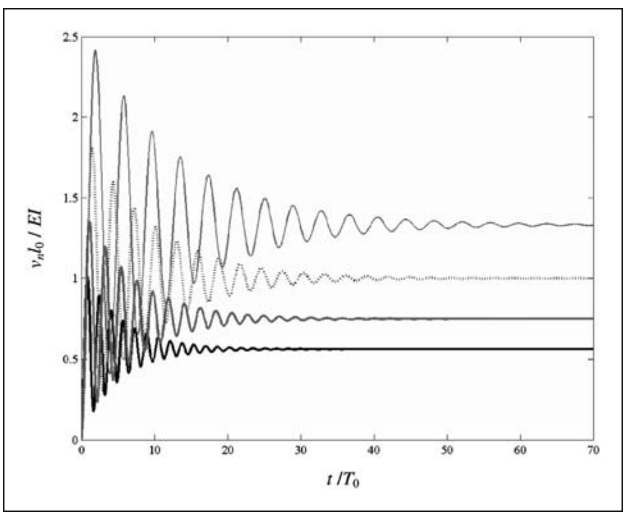

Figure 4. von Koch cantilever beam subjected to a unit step transversal force: dimensionless transversal displacement vs. normalized time, for different iterations n ($\theta = 60°$): $n = 2$ (thick black line), $n = 3$ (thick grey line), $n = 4$ (dotted line), $n = 5$ (grey line).

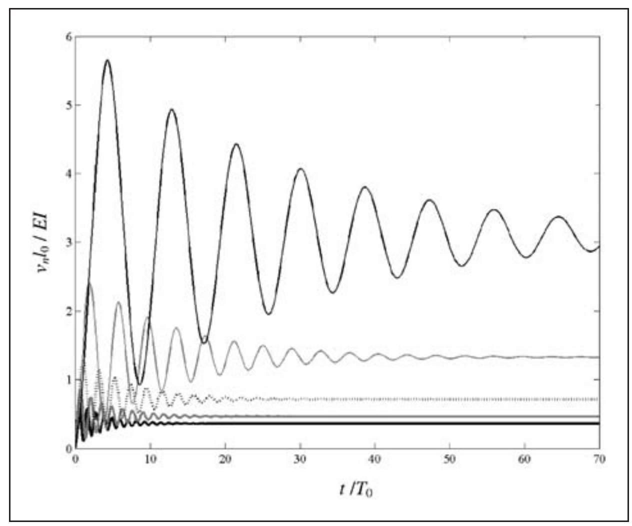

Figure 5. von Koch cantilever beam subjected to a unit step transversal force: dimensionless transversal displacement vs. normalized time, for different indentation angles θ ($n = 5$): $\theta = 15°$ (thick black line), $\theta = 30°$ (thick grey line), $\theta = 45°$ (dotted line), $\theta = 60°$ (grey line), $\theta = 75°$ (black line).

First of all, let us turn our attention to a specified von Koch cantilever beam, with a fixed angle θ, and let us consider the time history of the transversal displacement v_n at the free end for different iterations. Results are presented in Figure 4 (T_0 is the period related to the case of a rectilinear cantilever beam, n = 0): as the iteration n increases, the frequency of oscillation decreases, nevertheless its amplitude increases. Furthermore, the structure becomes more compliant, in perfect agreement with the analysis performed in (Carpinteri et al., 2009). Note that the steady-state response is reached earlier by lower order von Koch structures.

Let us now compare the step response between generic von Koch beams, with n fixed (Figure 5, $n = 5$): the frequency of oscillation decreases as the indentation angle θ increases, since lower natural frequencies correspond to higher values of θ (Carpinteri, 1997) (Figure 2). The steady-state response is reached earlier by smaller indentation angle structures, which are also resultingly stiffer: as a matter of fact, since the steady-state transversal displacement is inversely proportional to the structural stiffness, lower displacements correspond to stiffer structures.

Analogous results have been obtained by applying both a longitudinal unit step function and a unit impulse function. In the former case, the steady axial displacement at the free end is nearly an order of magnitude less than in the previous case, while in the latter the steady response is simply $v_n = 0$ (Figures 6 and 7). By summarizing, more damped responses are expected by either decreasing n or θ: these results, together with the multi-frequency analysis performed in Cohen (1996) and Puente et al. (2000) could be particularly useful in fractal antenna design.

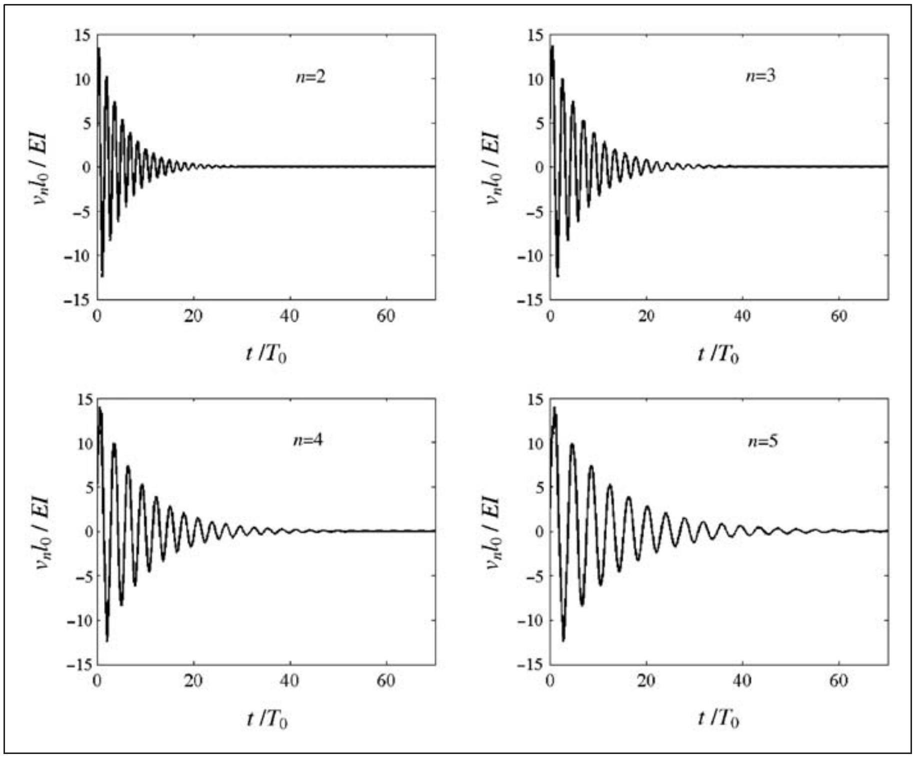

Figure 6. von Koch cantilever beam subjected to a unit impulse transversal force: dimensionless transversal displacement vs. normalized time, for different iterations n ($\theta = 60^\circ$).

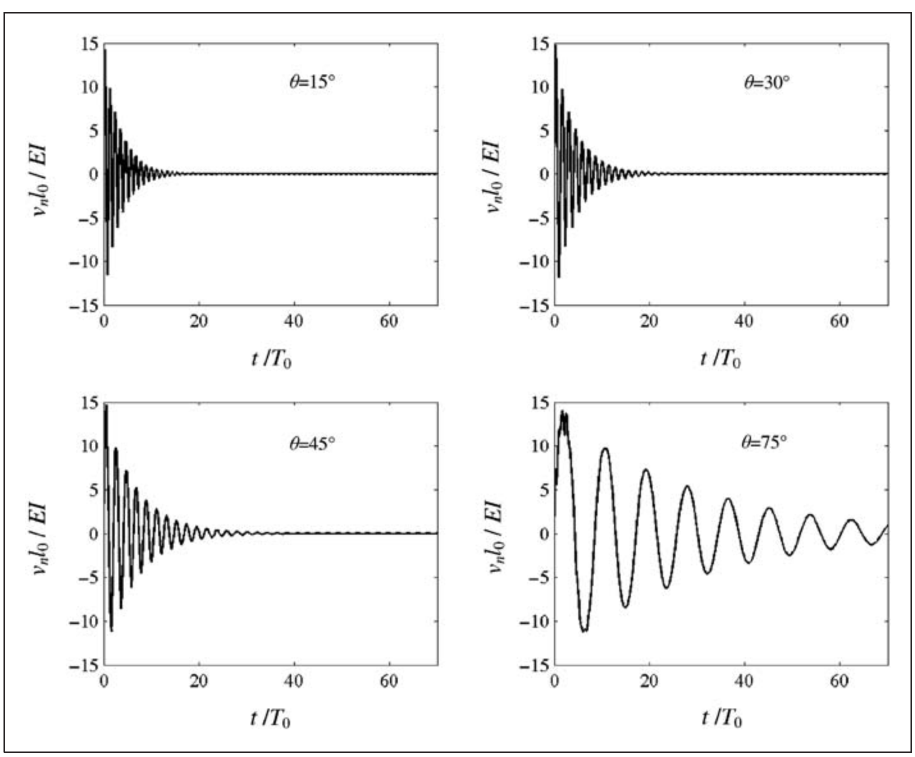

Figure 7. von Koch cantilever beam subjected to a unit impulse transversal force: dimensionless transversal displacement vs. normalized time, for different indentation angles θ ($n = 5$).

5. Conclusions

The mechanical behavior of a damped von Koch beam has been analyzed in this paper. By keeping fixed at each iteration n the number of master nodes, to which stiffness, mass and proportional damping matrices are reduced, simple recursive scaling laws are obtained. Eventually, the forced damped response of the structure for different iterations and indentation angles is investigated and compared: a methodology to analyze damping of fractal antennas is proposed.

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