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Free vibration analysis of a von Koch beam

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ABSTRACT

In this paper, the free undamped motion of a cantilever von Koch beam is investigated. The reduction of the stiffness and mass matrices leads to simple analytical recursive relationships depending on the fractal dimension of the structure. Results are then extended to perform a detailed modal analysis, which suggests peculiar scaling laws for the natural frequencies and modal shapes of the structure. Energy considerations are also provided. Finally, the potentiality of the von Koch beam as a fractal antenna is examined in terms of resonant frequencies.

1. Introduction

Several natural multiscale materials and structures have been recognized to possess a fractal-like geometry (Carpinteri, 1994a,b; Pugno, 2006). Moreover, the potentiality of fractal shapes for improving the efficiency of common man-made objects has recently been suggested in diverse research fields (Cohen, 1996; Lakes, 1995). Fractal-shaped antennas, for instance, have some unique characteristics that are linked to the geometrical properties of fractals (Cohen, 1996). Firstly, because the self-similarity property, which means that the structure is composed by some parts having the same shape as the whole but at a different scale, makes fractals specially suitable to design multi-frequency antennas (Gianvittorio and Rahmat-Samii, 2000; Puente et al., 2000): loosely speaking, this peculiar behaviour is due to the superposition action of the size-scaled constituents. Secondly, because the huge space-filling properties of some fractal shapes (i.e. the fractal dimension) help the realization of small antennas to better take advantage of the surrounding space (Wheeler, 1947).

Therefore, understanding the mechanical behaviour of fractal structures emerges as a primary topic. The main research programmes, in the framework of the mechanics of fractal solids, are based on the same approach: the solution on the fractal is found as the limit of the solutions obtained on the pre-fractals. Among these works, it is worthwhile mentioning those based on the local fractional calculus (Carpinteri et al., 2001; Carpinteri and Cornetti, 2002), according to which standard derivatives are replaced by fractional ones associated with the fractal dimension of the structure, and those based on the functional analysis, where the energy form convergence, which recovers the physical meaning of a Laplacian, is taken into account (Kigami, 2001; Mosco, 2002). Although different attempts exploiting the integration with respect to the Hausdorff measure have been made (Epstein and Śniatycki, 2006), the approach based on pre-fractals finds its validation on the fact that the fractal microstructure, especially in man-made objects, is generally exhibited only over a finite range of scales (Lakes, 1995; Pugno, 2006).

In this paper the attention is focused on the dynamic behaviour of a von Koch beam (von Koch, 1906). The von Koch beam is one of the most well-known fractals and its properties have largely been studied (see, for instance, Milosǒevic and Ristanovic, 2007 and related references). The structure turns out to be particularly suitable to design fractal antennas (Puente et al., 2000; Vinoy et al., 2002). Its static mechanical behaviour has recently been investigated in depth, both numerically, by means of a self-similarity postulate (Epstein and Adeeb, 2008), and analytically, by means of simple recursive relationships on the strain energy and stiffness matrix (Carpinteri et al., 2009).

The paper is structured as follows: the results obtained in (Carpinteri et al., 2009) on the strain energy and stiffness matrix are briefly resumed (Section 2). The mass matrix is then derived (Section 3), in the spirit of the Finite Element Method (FEM). Thanks to matrix reduction (Guyan, 1965), simple recursive scaling laws are provided. The free undamped motion of the structure is examined. After performing a modal analysis (Section 4), with particular interest in the fundamental frequencies (Section 4.1) and modal shapes (Section 4.2), the energy scaling is evaluated. Finally, the resonant frequencies of a von Koch beam are analyzed in detail and compared with those of a rectilinear beam (Section 5), showing the advantages in using fractals in antenna design.

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2. Stiffness matrix

The linear elastic deformation of a von Koch cantilever beam (Fig. 1), in the small deformations regime, has been analyzed in (Carpinteri et al., 2009). The structure has been loaded at the free end by the three elementary loadings: couple, transversal or longitudinal force. For each case, the recursive scaling of the strain energy Φ_n at the iteration n has been provided analytically. By neglecting the axial and shear compliances and referring to the strain energy Φ_1 related to the first order von Koch cantilever beam, these laws can be synthesized as:

$$\Phi_n = \frac{3}{4}\left(\frac{l_n}{l_0}\right)^{1-D} a_n^{(\Phi)}\Phi_1, \tag{1}$$

where l_0 is the length of the *initiator* $(n = 0)$, $l_n = l_0/3^n$ is the length of each of the 4^n segments constituting the structure at the nth step, $D = \ln 4/\ln 3$ is the fractal dimension of the von Koch curve. Coefficients $a_n^{(\Phi)}$ are equal to 1 in the case of the applied couple and rapidly converge to finite values in the cases of transversal or longitudinal forces (Carpinteri et al., 2009). Φ_1 is a function of the loading, of l_0, and of the beam rigidity $k = EI$, i.e. the product of the Young's modulus E of the material times the moment of inertia I of the cross-section with respect to the neutral axis. Note that, in order to have a simple unique scaling law (Eq. (1)), not the rectilinear beam $(n = 0)$ but the first order von Koch beam $(n = 1)$ is assumed as the reference iteration. As a matter of fact, it is possible to rewrite Eq. (1) with respect to the iteration $n = 0$ only in the cases of the couple or the transversal force, by simply eliminating the factor 3/4 and replacing Φ_1 with Φ_0 and $34/27\Phi_0$, respectively (Carpinteri et al., 2009). On the other hand, to what concerns the case of a longitudinal force applied to a rectilinear beam, the resulting strain energy Φ_0 would be equal to zero if the axial compliances are negligible.

The hypothesis of neglecting both axial and shear compliances is valid as long as the aspect ratio of each beam constituting the structure is slender. Since the length of each segment decreases as n increases (Fig. 1), some considerations are now necessary. From a practical point of view, for natural and (especially) man-made objects, the number of fractal iterations is generally low (for a fractal antenna, $n \leqslant 5$). In order to keep the validity of the assumption made above for all the iterations investigated, it is sufficient to consider a very slender initiator (say, $l_0/t_0 \approx 10^2 - 10^3$, where t_0 is the thickness). On the other hand, from a theoretical point of view, we can ideally start to consider a fractal beam at the generic iteration n, to model consistently the cross-section dimensions and to analyze the structure behaviour backwards. Since n is arbitrary, the hypothesis of neglecting axial and shear compliances could be extended also for n tending to infinite.

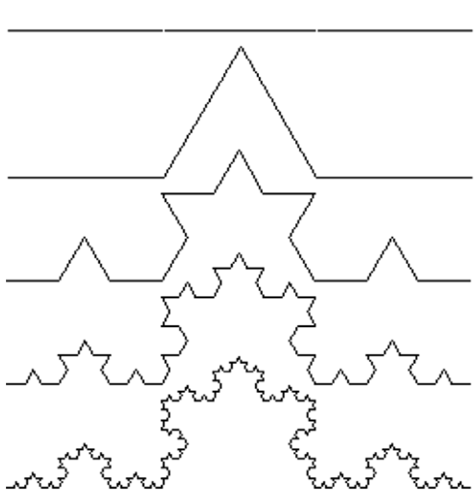

Fig. 1. First four iterations in the von Koch beam generation. At each step the middle third of each segment is removed and replaced by the other two sides of the equilateral triangle based on the removed segment (Feder, 1988). The fractal dimension of such a structure is $D = \ln 4/\ln 3$.

Results have then been extended for the computation of the stiffness matrix $[K]_n$ of the generic n-order of a von Koch beam (Fig. 2). Each beam e_j $(j = 1, \dots, 4^n)$ constituting the structure has been assumed to be a Euler–Bernoulli beam (for which cross-sections are assumed to remain plane and perpendicular to the deformed beam axis). Elements have been isolated and considered constrained by built-in supports in the end sections (Fig. 2a). By imposing the three generalized displacements of each support (rotations are assumed positive if counter-clockwise) and evaluating the corresponding reactions at both the ends, the local stiffness matrix of each element has been obtained.

Local matrices have then been rotated into the global reference system xy, assembled and condensated with respect to the six degrees of freedom (called master d.o.f.'s) of its ends (Guyan, 1965), numbered in the following order: rotation and y- and x-displacements of the left end, φ_L, v_L and w_L, rotation and y- and x-displacements of the right end, φ_R, v_R and w_R (Fig. 2). The following scaling law has been obtained (Carpinteri et al., 2009):

$$[K]_n = \frac{4}{3}\left(\frac{l_n}{l_0}\right)^{D-1} \frac{k}{l_0^3}[\bar{K}]_n, \tag{2}$$

Note that, in order to achieve dimensional homogeneity, the rotational variables are multiplied by the length l_0 and the moment variables are divided by the same length.

The matrix of the stiffness coefficients $[\bar{K}]_n$ is provided by the following expression:

$$[\bar{K}]_n = \begin{bmatrix} a_n^{(K)} & \frac{b_n^{(K)}}{2} & -d_n^{(K)} & \frac{b_n^{(K)}}{2} - a_n^{(K)} & -\frac{b_n^{(K)}}{2} & d_n^{(K)} \\ \frac{b_n^{(K)}}{2} & b_n^{(K)} & 0 & \frac{b_n^{(K)}}{2} & -b_n^{(K)} & 0 \\ -d_n^{(K)} & 0 & c_n^{(K)} & d_n^{(K)} & 0 & -c_n^{(K)} \\ \frac{b_n^{(K)}}{2} - a_n^{(K)} & \frac{b_n^{(K)}}{2} & d_n^{(K)} & a_n^{(K)} & -\frac{b_n^{(K)}}{2} & -d_n^{(K)} \\ -\frac{b_n^{(K)}}{2} & -b_n^{(K)} & 0 & -\frac{b_n^{(K)}}{2} & b_n^{(K)} & 0 \\ d_n^{(K)} & 0 & -c_n^{(K)} & -d_n^{(K)} & 0 & c_n^{(K)} \end{bmatrix} \tag{3}$$

where, because of the symmetry and equilibrium, the number of independent coefficients reduces to four: $a_n^{(K)}$, $b_n^{(K)}$, $c_n^{(K)}$, $d_n^{(K)}$. The values of the independent coefficients related to the first six von Koch beam iterations are presented in Table 1. As it can be seen, all the coefficients converge after the first six iterations. Hence, the stiffness matrix $[K]_n$ in Eq. (2) scales asymptotically exactly as $(3/4)^n$ or, equivalently, as $(l_n/l_0)^{D-1}$. From a theoretical point of view, for n tending to infinity the structural stiffness trivially tends to zero and the structure becomes infinitely compliant. If the beam rigidity $k_n = E_n I_n$ is supposed to scale at each iteration as:

$$k_n = \left(\frac{4}{3}\right)^{n-1} k_1 = \frac{3}{4}\left(\frac{l_n}{l_0}\right)^{1-D} k_1, \tag{4}$$

the stiffness matrix $[K]_n$ remains finite as n increases (Eq. (2)).

Note that at each step the structure can be regarded as the union of four identical copies of the structure at the previous step, each reduced by a factor 3. The similarity principle introduced in (Epstein and Adeeb, 2008), according to which the stiffness matrix of a reduced copy of a fractal is proportional to the stiffness matrix of the original fractal (Fig. 2b), is proved in our analysis and can be exploited, equivalently, to the evaluation of the global stiffness matrix.

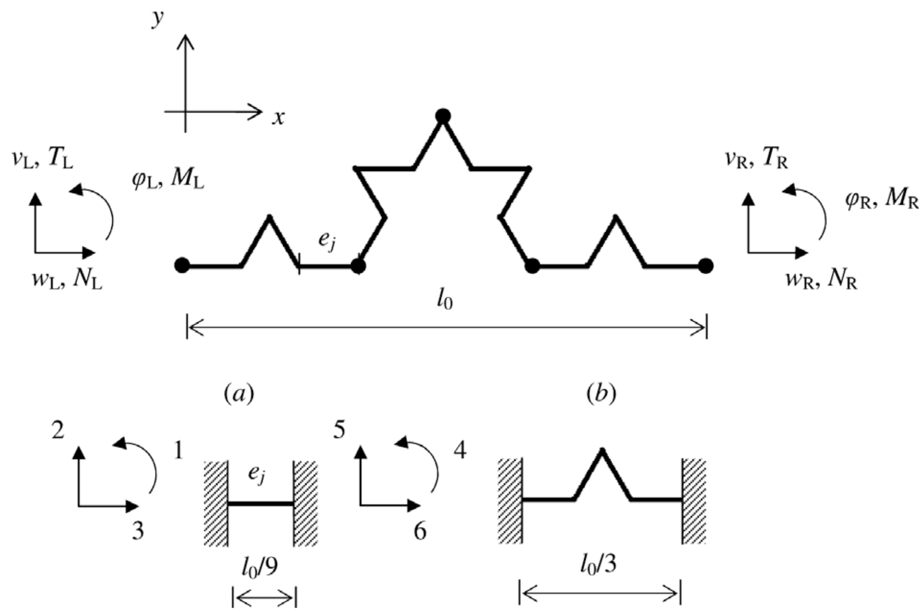

Fig. 2. von Koch beam, $n = 2$: degrees of freedom at the ends. The structure is composed by 16 beams of length $l_2 = l_0/9(a)$ or, equivalently, by four reduced copy of the fractal at level $n = 1(b)$. Black dots represent the main vertices of the first order von Koch beam.

Table 1
Stiffness coefficients of the matrix $[\overline{K}]_n$ related to the first six iterations of the von Koch beam.

Iteration, n	1	2	3	4	5	6
$a_n^{(K)}$	4.0928	4.3927	4.4783	4.4974	4.5018	4.5031
$b_n^{(K)}$	11.5714	11.3922	11.3021	11.2703	11.2610	11.2598
$c_n^{(K)}$	86.4000	97.6889	100.6736	101.2127	101.3058	101.3149
$d_n^{(K)}$	6.2354	8.8089	9.5333	9.6993	9.7379	9.7465

3. Mass matrix

In the dynamics analysis, once the stiffness matrix is obtained, the problem turns into how to generate the mass matrix. The simplest way of generating the element mass matrix is by lumping the mass at the nodes: an equivalent mass is considered in correspondence to each nodal translation, while the mass coefficients corresponding to the rotational coordinates are zero. Each equivalent mass could be evaluated, for example, by adding the weights, divided by two, of the beams which converge into the node. By operating in such a way, the corresponding mass matrix results to be diagonal with clear advantages from the numerical computation.

On the other hand, this approach has several drawbacks: lumping is an arbitrary process, so that some control over the error involved in discretization is lost. More serious is the fact that lumping can lead to singular mass matrices, which is at odds with the fact that mass matrices are positive definite by definition. As an alternative, the mass matrix could be derived by the same approach as that of the stiffness matrix (i.e. the Finite Element Method, FEM), taking into account the real distribution of the masses over the beam. In such a case, the mass matrix is said to be consistent.

In this analysis, the mass matrix will be generated by means of the second approach (FEM). The consistent mass matrix obtained by linear interpolation functions (for details see Carpinteri, 1997) and related to a beam of length l_0 and mass per unit length $m = \rho A$, where ρ is the material density and A is the area of the cross-section, is:

$$[M]_0 = \frac{\rho A l_0}{420}\begin{bmatrix} 4l_0^2 & 22l_0 & 0 & -3l_0^2 & 13l_0 & 0 \\ 22l_0 & 156 & 0 & -13l_0 & 54 & 0 \\ 0 & 0 & 140 & 0 & 0 & 70 \\ -3l_0^2 & -13l_0 & 0 & 4l_0^2 & -22l_0 & 0 \\ 13l_0 & 54 & 0 & -22l_0 & 156 & 0 \\ 0 & 0 & 70 & 0 & 0 & 140 \end{bmatrix}. \tag{5}$$

At each iteration of the von Koch beam, the mass matrix related to each element constituting the structure is again written and rotated into the global reference system xy (it can be proved, again, that the mass matrix of a reduced copy of a von Koch beam is proportional to the mass matrix of the original fractal (Fig. 2)). Matrices are then assembled and, by means of Guyan's reduction, they are condensated with respect to the six degrees of freedom of the two ends of the structure. The following relationship can be finally derived:

$$[M]_n = \left(\frac{l_n}{l_0}\right)^{1-D}\frac{ml_0}{560}[\overline{M}]_n, \tag{6}$$

where

$$[\overline{M}]_n = \begin{bmatrix} a_n^{(M)} & e_n^{(M)} & d_n^{(M)} & -g_n^{(M)} & m_n^{(M)} & l_n^{(M)} \\ e_n^{(M)} & b_n^{(M)} & f_n^{(M)} & -m_n^{(M)} & h_n^{(M)} & -o_n^{(M)} \\ d_n^{(M)} & f_n^{(M)} & c_n^{(M)} & l_n^{(M)} & o_n^{(M)} & -i_n^{(M)} \\ -g_n^{(M)} & -m_n^{(M)} & l_n^{(M)} & a_n^{(M)} & -e_n^{(M)} & d_n^{(M)} \\ m_n^{(M)} & h_n^{(M)} & o_n^{(M)} & -e_n^{(M)} & b_n^{(M)} & -f_n^{(M)} \\ l_n^{(M)} & -o_n^{(M)} & -i_n^{(M)} & d_n^{(M)} & -f_n^{(M)} & c_n^{(M)} \end{bmatrix}. \tag{7}$$

The matrix has again been homogenized, for the sake of simplicity, as done in Section 2 for the stiffness matrix. The mass coefficients are presented in Table 2. It is evident that, after six iterations, each coefficient nearly converges. The mass matrix $[M]_n$ in Eq. (6) scales, hence, asymptotically exactly as $(4/3)^n$ or, equivalently, as $(l_n/l_0)^{1-D}$.

Eq. (6) represents the counterpart of Eq. (2), obtained for the stiffness matrix. While each term in the stiffness matrix tends to vanish as the number of iterations n increases (by scaling as $(3/4)^n$), the terms in the mass matrix diverge (by scaling as $(4/3)^n$).

The mass matrix remains constant only if the mass per unit length m_n scales as:

$$m_n = \left(\frac{3}{4}\right)^{n-1} m_1 = \frac{4}{3}\left(\frac{l_n}{l_0}\right)^{D-1} m_1, \tag{8}$$

in analogy with Eq. (4).

It is noteworthy to point out, at this point, that stiffness matrix reduction is exact, while mass reduction requires neglecting all inertia loads with respect to elastic loads.

Our choice of reduction the matrices with respect to the degrees of freedom of their ends has been made for the sake of simplicity, and will reveal to be sufficient to study the first two modes in the free vibration analysis of the beam (Section 4). What is important, since it represents the novelty of this paper, is the asymptotic behaviour of Eqs. (2) and (6), which has been proved not to change if the number of master degrees of freedom is increased, as deducible also from the self-similarity property. This aspect will allow to investigate, without loosing of generality, also higher modes, the resonant frequencies of which would be, otherwise, overestimated.

4. Undamped free vibration

The equation of motion of an N-degrees of freedom system, in absence of damping and of external forces, could be written in the following form:

$$[M]\{\ddot{\delta}\} + [K]\{\delta\} = \{0\}, \tag{9}$$

where $[M]$ and $[K]$ are the $N \times N$ mass and stiffness matrices respectively, and $\{\delta\}$ represents the N-dimensional vector of nodal displacements.

The solution of Eq. (9) is:

$$\{\delta\} = \{\delta_0\}\sin(\omega t - \alpha), \tag{10}$$

being $\{\delta_0\}$ the amplitude of displacement vector and α the phase difference.

Substituting Eq. (10) into Eq. (9), yields:

$$([K] - \omega^2[M])\{\delta_0\} = \{0\}, \tag{11}$$

which is known as the eigenvalue equation. Since the trivial solution lacks a physical meaning, the determinant of the system coefficients in Eq. (11) must be equal to zero:

$$\mathrm{Det}([K] - \omega^2[M]) = 0, \tag{12}$$

which represents the characteristic equation of degree N in ω^2, giving the eigenvalues. The square roots of these quantities are the *natural angular frequencies*, while the corresponding eigenvectors represent physically the so-called *natural modes of vibration* of the system.

Once the angular frequencies are known, it is easy to evaluate the natural frequencies and periods as:

$$f = \frac{\omega}{2\pi}, \tag{13a}$$

$$T = \frac{1}{f}. \tag{13b}$$

The lowest frequency f_1 is referred to as the *fundamental frequency*, and for different practical problems it is the most important.

Note that, in the linear undamped case, the natural frequencies given by Eq. (13a) coincide with the resonant frequencies.

4.1. Natural frequencies

The governing differential equation of motion of a von Koch beam, in its free natural vibration, can be written as:

$$[M]_n\{\ddot{\delta}\}_n + [K]_n\{\delta\}_n = \{0\}, \tag{14}$$

where $[M]_n$ and $[K]_n$ are the reduced matrix provided by Eqs. (2) and (6), respectively, $\{\delta\}_n$ is the vector of nodal displacements $(\{\delta\}_n = (\varphi_L l_0, v_L, w_L, \varphi_R l_0, v_R, w_R)_n^T$ if the extreme nodes are taken into account) and $\{\ddot{\delta}\}_n$ is the corresponding acceleration vector at iteration n.

In the case of a von Koch cantilever beam, by assigning boundary conditions, the related eigenvalue problem could be easily solved. Since the stiffness matrix scales as $(3/4)^n$ (Eq. (2)) and the mass one as $(4/3)^n$ (Eq. (6)), it is legitimate to expect the scaling law for the natural frequencies $f_{i,n}$ as (let us remind that $\omega \propto \sqrt{(k/m)}$):

$$f_{i,n} = \left(\frac{3}{4}\right)^{n-1} a_{i,n}^{(f)} f_{i,1} = \frac{4}{3}\left(\frac{l_n}{l_0}\right)^{D-1} a_{i,n}^{(f)} f_{i,1}, \tag{15}$$

where the first subscript refers to the mode, while the second one refers to the von Koch beam iteration ($f_{1,n}$, for instance, is the fundamental frequency related to the nth order iteration of the von Koch

Table 2
Coefficients of the matrix $[\overline{M}]_n$ related to the first six iterations of the von Koch beam.

Iteration, n	1	2	3	4	5	6
$a_n^{(M)}$	5.3040	4.8722	4.7354	4.7063	4.7016	4.6993
$b_n^{(M)}$	211.6056	215.2500	215.5875	215.6625	215.6834	215.6975
$c_n^{(M)}$	306.9144	289.9875	290.0220	289.9125	289.8230	289.7268
$d_n^{(M)}$	21.5815	13.692	11.6375	11.1140	10.9828	10.9455
$e_n^{(M)}$	30.8652	30.0371	29.4644	29.3107	29.2811	29.2659
$f_n^{(M)}$	115.9132	108.5840	107.2181	106.8162	106.6687	106.5980
$g_n^{(M)}$	3.1509	2.4241	2.2586	2.2209	2.2135	2.2113
$h_n^{(M)}$	68.3944	64.7400	64.4127	64.3624	64.3366	64.3285
$i_n^{(M)}$	26.9144	9.9884	10.0263	9.9123	9.8301	9.7531
$l_n^{(M)}$	12.6475	12.0721	11.8982	11.8467	11.8334	11.8233
$m_n^{(M)}$	14.9431	12.8233	12.3946	12.2998	12.281	12.2754
$o_n^{(M)}$	57.6363	54.8433	54.6985	54.5960	54.5280	54.4740

cantilever beam). Values of the adimensional coefficients $a_{i,n}^{(f)}$ are shown in Table 3. As can be seen, they converge after the first six iterations, the maximum percentage difference between the last two iterations being nearly 1‰.

Eq. (15) can be exploited to evaluate the vibration frequencies of the structure at the generic step n, once those at level 1 are known. On the other hand, it should be outlined that, although Guyan's reduction precision is guaranteed only when excitation frequency is close to zero, evaluated frequencies are normally satisfactory in the domain of $[0, 0.3f_s]$, where f_s is the smallest natural frequency of the structure, with all the master d.o.f.'s restrained (Bouhaddi and Fillod, 1994). In other higher domains, the errors of the results are larger and sometimes unacceptable.

In the dynamic analysis of a von Koch beam, if stiffness and mass matrices are condensated with respect to the d.o.f.'s of their ends, results are accurate for mode I, $f_{1,n}/(0.3f_{s,n}) \approx 0.4$ (the relative error is less than 3‰), and acceptable for mode II, $f_{2,n}/(0.3f_{s,n}) \approx 1.5$ (the relative error being nearly 2%). For higher modes, frequency values are overestimated by more than 50%: more sophisticated techniques (see (Savini and Vivo, 2007) and related references) or Guyan's reduction with respect to a larger number of degrees of freedom must hence be considered. Values reported in Table 3 corresponding to mode III are computed by choosing as masters the 15 d.o.f.'s related to the five nodes of the first order von Koch beam (indeed 12, since one end is considered restrained, Fig. 2). In this case $f_{3,n}/(0.3f_{s,n}) \approx 0.35$ (the relative error is less than 4‰) and evaluated frequencies are accurate up to seventh mode. Note that the asymptotic behaviour of Eq. (15) does not change by increasing the number of master d.o.f.'s, as proved by considering also subsequent modes.

Taking into account Eq. (15), it is then possible to write the scaling law related to the natural periods $T_{i,n}$ as:

$$T_{i,n} = \left(\frac{4}{3}\right)^{n-1} b_{i,n}^{(T)} T_{i,1} = \frac{3}{4}\left(\frac{l_n}{l_0}\right)^{1-D} b_{i,n}^{(T)} T_{i,1}, \tag{16}$$

where $b_{i,n}^{(T)} = 1/a_{i,n}^{(f)}$.

4.2. Modal shapes

Once the eigenvalues ω^2 of Eq. (12) are known, the corresponding eigenvectors $\{\delta_0\}$ could be easily evaluated. Let us remind that the shape of the natural modes is unique, but the amplitude is not and it is defined unless a multiplying constant. The process of adjusting the elements of natural modes to render their amplitude unique is called normalization, and the resulting vectors are referred to as normal modes. A very convenient normalization scheme consists of setting:

$$\{\delta_0\}_j^T [M] \{\delta_0\}_j = 1 \quad j = 1, 2, \ldots, N, \tag{17a}$$

which yields:

$$\{\delta_0\}_j^T [K] \{\delta_0\}_j = \omega_j^2 \quad j = 1, 2, \ldots, N. \tag{17b}$$

Clearly, by exploiting the results obtained in the previous sections (Eqs. (2), (6) and (15)), a recursive relationship could be easily obtained. Indeed, it is evident from Eqs. (17a,b) that the normalized eigenvectors will scale as $(3/4)^{n/2}$. For example, if the fundamental frequency is taken into account:

$$\{\delta_0\}_{1,n} = \sqrt{\frac{4}{3}} \left(\frac{l_n}{l_0}\right)^{\frac{D-1}{2}} [A_{1,n}] \{\delta_0\}_{1,1}. \tag{18}$$

The matrix $[A_{1,n}]$ is diagonal and could be expressed in terms of its coefficients as:

$$[A_{1,n}] = \begin{bmatrix} a_{1,n}^{(\delta)} & 0 & 0 \\ 0 & b_{1,n}^{(\delta)} & 0 \\ 0 & 0 & c_{1,n}^{(\delta)} \end{bmatrix}. \tag{19}$$

Their values are presented in Table 4, starting from the values reported in Tables 1 and 2. Again, if the first four accurate digits are taken into account, convergence is achieved within the first six iterations, the maximum percentage difference between the last two iterations being nearly 1‰.

Eqs. (18) could be generalized by considering the subsequent frequencies:

$$\{\delta_0\}_{i,n} = \sqrt{\frac{4}{3}} \left(\frac{l_n}{l_0}\right)^{\frac{D-1}{2}} [A_{i,n}] \{\delta_0\}_{i,1}, \tag{20}$$

where the coefficients corresponding to the third modal shape have been evaluated by increasing the number of master d.o.f.'s to 12 (Table 4), in accord with what performed on the related frequency. Note that, for the sake of simplicity, only the coefficients related to the extreme node have been reported. Clearly, the more the number of master d.o.f.'s, the more information on how $\{\delta_0\}_{i,n}$ varies along the length of the beam are provided. A direct comparison, as that expressed by Eq. (20), with the modes of a rectilinear beam is not straightforward, since, under the hypothesis made in Section 2, this structure possesses an infinite rigidity in the x direction.

Let us now introduce the *modal matrix* $[\Delta]$, whose columns are the evaluated eigenvectors, similarly to Eqs. (17a,b). The following relationships keep true:

$$[\Delta]^T [M] [\Delta] = [I], \tag{21a}$$

$$[\Delta]^T [K] [\Delta] = [\omega^2], \tag{21b}$$

where $[I]$ is the unit matrix and $[\omega^2]$ the diagonal matrix of the natural angular frequencies. Then, Eq. (20) could be rewritten in the form:

$$[\Delta]_n = \sqrt{\frac{4}{3}} \left(\frac{l_n}{l_0}\right)^{\frac{D-1}{2}} [\overline{\Delta}]_n, \tag{22}$$

where the matrix $[\overline{\Delta}]_n$ is expected to converge after a few iterations.

The modal deformations related to a third order von Koch cantilever beam obtained by means of LUSAS® code are reported in Fig. 3.

Table 3
Coefficients $a_{i,n}^{(f)}$ related to the natural frequency scaling law (Eq. (15)).

Iteration, n	1	2	3	4	5	6
$a_{1,n}^{(f)}$	1	0.9968	0.9936	0.9921	0.9919	0.9918
$a_{2,n}^{(f)}$	1	1.0994	1.1169	1.1203	1.1210	1.1212
$a_{3,n}^{(f)}$	1	1.0226	1.0251	1.0233	1.0227	1.0225

Table 4
First six iterations of a von Koch cantilever beam: coefficients related to the normal mode scaling law (Eq. (20)).

Iteration, n	1	2	3	4	5	6
$a_{1,n}^{(\delta)}$	1	0.9861	0.9820	0.9804	0.9799	0.9799
$b_{1,n}^{(\delta)}$	1	0.9935	0.9910	0.9901	0.9898	0.9898
$c_{1,n}^{(\delta)}$	1	1.2939	1.3601	1.3767	1.3809	1.3821
$a_{2,n}^{(\delta)}$	1	1.1013	1.1143	1.1164	1.1167	1.1167
$b_{2,n}^{(\delta)}$	1	1.0786	1.0826	1.0821	1.0818	1.0819
$c_{2,n}^{(\delta)}$	1	1.1177	1.1407	1.1464	1.1477	1.1481
$a_{3,n}^{(\delta)}$	1	1.0763	1.0747	1.0743	1.0741	1.0741
$b_{3,n}^{(\delta)}$	1	1.0936	1.0921	1.0912	1.0911	1.0910
$c_{3,n}^{(\delta)}$	1	0.6986	0.6287	0.6123	0.6088	0.6080

Finally, some basic energy considerations are necessary. The potential and kinetic energies associated with each mode are defined by the following relationships, respectively:

$$\mathscr{W} = \frac{1}{2}\{\delta\}_j^T[K]\{\delta\}_j, \tag{23a}$$

$$\mathscr{T} = \frac{1}{2}\{\dot{\delta}\}_j^T[M]\{\dot{\delta}\}_j, \tag{23b}$$

where the vector of nodal displacements $\{\delta\}$ has been defined in Eq. (10). Both $\mathscr{W}$ and $\mathscr{T}$ scale as $(3/4)^{2n}$, as it is evident from Eqs. (11) and (20). On the other hand, if the scaling laws on the beam rigidity and mass per unit length of Eqs. (4) and (8) are taken into account, the potential and kinetic energies remain finite and different from zero (Eqs. (23a,b)).

Results could be extended to the analysis of a von Koch beam with a different indentation angle θ (in this paper it has always been considered $\theta = 60^\circ$). In that case, a different fractal dimension D must be considered (Vinoy et al., 2002):

$$D = \frac{\ln 4}{\ln 2(1+\cos\theta)}. \tag{24}$$

Note that, if damping has to be considered, Guyan's reduction also applies to the damping matrix (work in progress).

5. Resonant frequencies of a von Koch antenna

In recent years, fractal geometries have been studied for antenna design. As outlined in Section 1, the self-similarity of fractal geometries is strictly linked to the multi-frequency characteristics (Vinoy et al., 2002). Moreover, another important fractal characteristic is its huge space filling (Feder, 1988): certain monopoles could hence be designed to have an arbitrarily large surface, although they can be constrained to fit a given volume. Therefore, it is possible to design small fractal antennas having huge surface (Puente et al., 2000). These properties are investigated in this section, taking into account that the von Koch beam has been found to be suitable for antenna design.

In Section 4.1 the natural frequencies of a cantilever von Koch beam have been evaluated with particular emphasis to the fundamental one and a peculiar scaling law has been derived (Eq. (15)). It should again pointed out that, in the linear undamped case, these frequencies coincide with the resonant ones. Cleary, Eq. (15) already shows that the scaling law of the resonant frequencies is strictly connected with the fractal dimension of the structure. Moreover, as a consequence of the scaling, the ratios $f_{i+1,n}/f_{i,n}$ reasonably converge after the first six iterations. This behaviour is in good agreement with that observed for the von Koch dipole in (Vinoy et al., 2002): anyway, in that work, the convergence is not found and it is concluded that "the ratios remain nearly the same" (note, however, that only the first two accurate digits are considered).

Furthermore, let us now compare the resonant frequencies of a von Koch cantilever beam with those of a rectilinear beam having the same cross-section dimensions. The total length L_n of a von Koch beam is defined at each iteration as (Fig. 1):

$$L_n = 4^n l_n = (4/3)^n l_0. \tag{25}$$

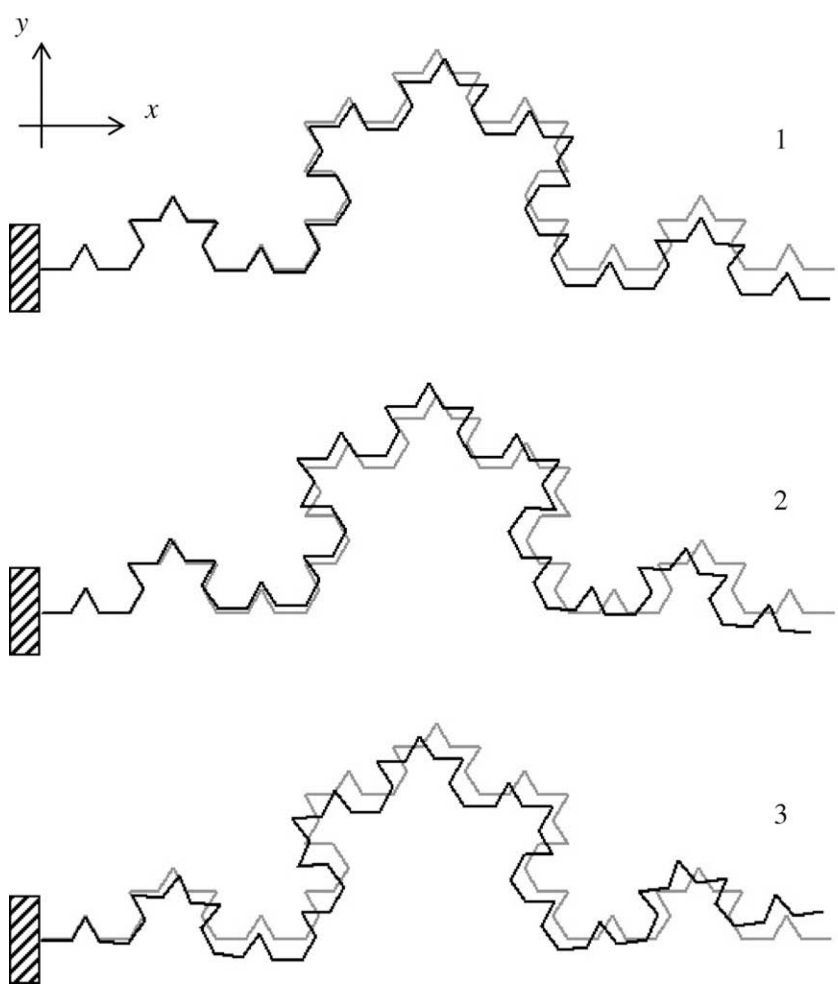

Fig. 3. von Koch cantilever beam, $n = 3$: first three modal shapes.

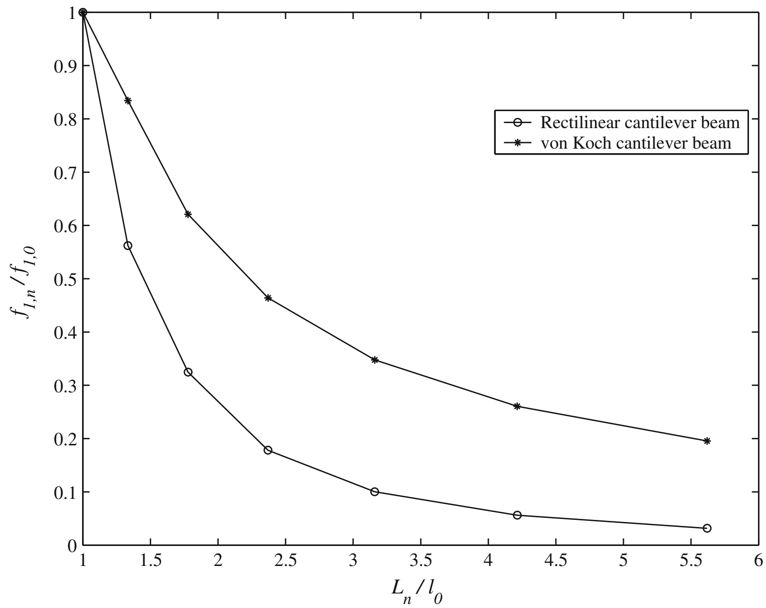

Fig. 4. First resonance frequency vs. total length for a rectilinear cantilever beam and for different iterations n of a von Koch cantilever beam with initiator length l_0.

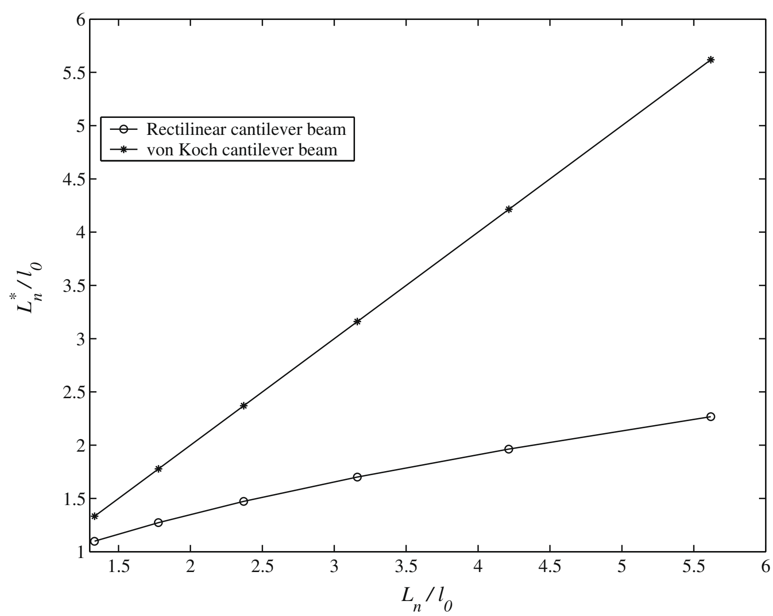

Fig. 5. Length L_n^* that a cantilever beam should have to possess the same fundamental frequency of a von Koch cantilever beam, whose initiator length is l_0, for different iterations n.

In Fig. 4 the fundamental resonance frequency is plotted as a function of the total length for a rectilinear cantilever beam and a von Koch cantilever beam, whose initiator length is l_0, respectively. As it can be noticed, the frequencies related to a rectilinear beam are lower and converge faster to zero. Let us now denote with L_n^* the length that a rectilinear cantilever beam should have to possess the identical fundamental frequency of a given von Koch beam (l_0 fixed). As can be seen from Fig. 5, L_n^* must clearly increase as n increases. Its values exceed that of the fractal initiator ($L_n^*/l_0 \approx 2$ for $n = 5$) and it does not grow at the same rate as the length L_n defined in Eq. (25). From the opposite point of view, the length of the initiator, that a von Koch beam should have to be resonant at the same fundamental frequency of a straight beam, decreases at each iteration, while the total length increases. In synthesis, the fractal antenna is physically smaller and has a larger lateral surface (Gianvittorio and Rahmat-Samii, 2000; Puente et al., 2000). Eventually note that, differently from the results obtained in (Puente et al., 2000), no saturation point (i.e., no horizontal asymptote) is expected as n increases, consistently with the analysis performed in the previous sections.

6. Conclusions

The dynamic behaviour of a von Koch cantilever beam, in absence of damping and of external forces, has been investigated in this paper. Scaling laws of the reduced stiffness and mass matrices lead to the investigation of the structure free vibration analysis, with particular emphasis to the fundamental frequencies and modal shapes. Simple recursive relationships emerge, which have been found in an excellent agreement with numerical simulations. Eventually, the resonant frequencies of a von Koch cantilever beam are compared with those of a rectilinear one. Results have demonstrated to be useful in fractal antenna design.

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