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Scaling laws and fractality in the framework of a phenomenological approach

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ABSTRACT

Phenomenological Universality (PUN) represents a new tool for the classification and interpretation of different non-linear phenomenologies in the context of cross-disciplinary research. Also, they can act as a "magnifying glass" to finetune the analysis and quantify the difference among similarly looking datasets. In particular, the class *U*2 is of special relevance since it includes, as subcases, most of the commonly used growth models proposed to date. In this contribution we consider two applications of special interest in two subfields of Elasto-dynamics, i.e. Fast- and Slow-Dynamics, respectively. The results suggest that new equations should be adopted for the fitting of the experimental results and that fractal-dimensioned variables should be used to recover the scaling invariance, which is invariably lost due to non-linearity.

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1. Introduction

Scaling is a very powerful tool for the quest of universal laws in all sciences. In fact, in microphysics, from leptons to molecules, we run the gamut of approximately 25 orders of magnitude (in volume). In "ordinary physics", to arrive, e.g., to the volume of the earth, we require another 46 orders of magnitude, and in astrophysics another 52, to reach the estimated volume of the Universe. Likewise, in Biology, life processes cover more than 27 orders of magnitude (in mass) from gene molecules to whales and giant sequoias.

Scaling laws [1] can be our Ariadne's thread through those awesome ranges, since they are manifestations of intrinsic mechanisms (such as energy conservation, phase transitions, complexity and/or randomness), which may be basically the same even in very different fields. Almost paradoxally, in an allometric analysis, the investigation of broken scaling symmetries may lead one to unexpected momentous discoveries, as it is often the case with broken symmetries (the non-conservation of parity in weak interactions conferred The Nobel Prize to T.D. Lee and C. N. Yang, 1957).

The exploitation of scaling laws in the contest of energy balance in living organisms has led G. B. West and collaborators to a sequel of very elegant results [2–5], among them the conjectures of a universal growth law for all living organisms, of natural selection evolved hierarchical fractal branching networks and of a "biological clock" for individual species. Their model of a universal growth curve has been later extended to neoplasies [6–8] and, even more recently, to a completely cross-disciplinary range of problems in the context of a Phenomenological Universalities (PUN) approach [9,10], as discussed briefly in the following Section.

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2. Phenomenological universalities

Mathematical universalities, as provided e.g. by partial differential equations, represent the only "true" universalities. In a "top–down" approach they have been used for centuries. However, often we are confronted with observational or experimental datasets: the task is to "infer" from them some more or less general "laws" in a "bottom–up" approach. PUN's represent a paradigm to perform such a task at a most general level.

In order to explain the PUN approach from an applicative point of view, let us assume that we have an experimental dataset: $y_i = y(t)$, where t can be the time (or any other independent variable) and y any observable depending on it. The usual procedure is to perform a fitting of the data, but the choice of the fitting function is generally arbitrary. As a result the analysis is, in general, only qualitative, and often based on the visual inspection of the plots. By contrast we wish to proceed here in a way that is justified by a "universal" approach, i.e. totally independent of the field of application.

If the nature of the problem suggests that it can be reduced to a first order ODE, we aim to analyse it starting from the non-linear growth equation

$$\dot{y}(t) = a(y,t)y(t) \tag{1}$$

where $\dot{y} = \frac{dy}{dt}$, and a represents the growth rate. Eq. (1) is, of course, not limited to the modeling of growth problems, since there is no restriction on the nature of the variables y and t. Its generality may be enhanced by putting

$$z = \ln(y) \tag{2}$$

Then Eq. (1) may be rewritten as

$$\dot{z} = a(z,t). \tag{3}$$

However, Eq. (1) or (3) in their generality cannot take us too far. In order to use them for a quantitative analysis, it is necessary to restrict its scope by means of some 'constraints', which, although arbitrary, at least are independent on the particular field of application.

Let us then assume that a is a function solely of t and that its derivative with respect to z may be expanded as a set of powers of a. It follows

$$b = \dot{a} = \frac{da}{dz}\dot{z} = \sum_{n=1}^{\infty} \alpha_n a^n(z). \tag{4}$$

If a satisfactory fit of the experimental data is obtained by truncating the set at the N-th term (or power of a), then we state that the underlying phenomenology belongs to the Universality Class UN.

It can be easily shown that the universality class $U1$ (i.e. with $N = 1$) represents the well known 'Gompertz' law [11], which has been used for more than a century to study all kinds of growth phenomena. The class $U2$ includes, besides Gompertz as a special case, all the growth models proposed to date in all fields of research, i.e., besides the already mentioned model of West and collaborators [4,5], the exponential, logistic, theta-logistic, potential, von Bertalanffy, etc. (see for a review Ref. [12]).

By solving the differential equations $\dot{z} = a$ and $\dot{a} = b$, with b written, for brevity (in the case $N = 2$)

$$b = \alpha a + \beta a^2, \tag{5}$$

we find the $U2$ normalized solution

$$y = \{1 + \beta/\alpha[1 - \exp(\alpha t)]\}^{(-1/\beta)}. \tag{6}$$

It is interesting to observe that Eq. (6) can be written as

$$u = c_1 + c_2\tau, \tag{7}$$

which shows that the scaling invariance, which was lost due to the non-linearity of $a(z)$, may be recovered if the fractal-dimensioned variable $u = y^{-\beta}$ and $\tau = \exp(\alpha t)$ are considered [13,14]. In fact β is, in general, non-integer. In Eq. (7) c_1 and c_2 are constants: $c_2 = -\beta/\alpha, c_1 = 1 - c_2$.

It may also be useful to note that y is the solution of the ODE

$$\dot{y} = \gamma_1 y^p - \gamma_2 y, \tag{8}$$

where $p = 1 + \beta; \gamma_1$ and γ_2 are two constants: $\gamma_2 = \alpha/\beta$ and $\gamma_1 = 1 - \gamma_2$. Their sum is equal to 1, due to the chosen normalization ($y(0) = 1$). Eq. (8) coincides with West's universal growth equation [5], except that here p may be totally general, while West and collaborators adopt Kleiber's prescription ($p = 3/4$) [15], which seems to be well supported by animal growth data. For other systems different choices of p may be preferable: in particular C. Guiot et al. suggest a dynamical evolution of p in the transition from an avascular phase to an angiogenetic stage in tumors [16]. Eq. (8) has a very simple energy balance interpretation, with $\gamma_2 y^p$ representing the input energy (through a fractal branched network), $\gamma_2 y$ the metabolism and $\dot{y}$ the asymptotically vanishing growth.

3. Applications and results

As an example of application of the proposed methodology, we consider in the following two different problems of interest in the fields of the elasto- and thermo-elasto-dynamics. Further examples of universality, in the fields of Biology, Oncology and Auxology, respectively, are dicussed elsewhere [1,7,17].

Let us consider, at first, the case of a quasi-static experiment, obtained by numerical simulations on a given sample (e.g. concrete) with bulk modulus $K = 25$ GPa, by cyclical variations of the stress, applied in a uni-axial compressional experiment. A hysteretic loop may be observed in the strain vs. stress curve, i.e. the values of the strain are different during the loading and unloading for the same values of the stress. This is a well known [18] effect, which can be more or less conspicuous, depending on the material under consideration.

Assuming to be in saturation conditions, i.e. that both the loading and unloading curves close up in the origin, where we have set both stress and strain be equal zero, it is convenient to use directly Eq. (3) with t being the applied stress and z the corresponding strain. The solution can then be immediately derived by Eqs. (4) and (2):

$$z = -\frac{1}{\beta}\ln\{1 + \beta a_0/\alpha[1 - \exp(\alpha t)]\}, \tag{9}$$

where a_0 represents the value of $a(z)$ in the origin.

The fitting obtained (see Fig. 1) is excellent ($R^2 \approx 1$), confirming that this phenomenology belongs to the class $U2$. For convenience in the comparison, synthetic data have been used [19], rather then data from actual experiments, since in several previous works it has been proved that simulations of quasi-static experiments using a PM space model [20] yield an excellent level of agreement with the latter [21].

An immediate corollary to our result is that Eq. (9) is integrable, giving as a result

$$h = -\frac{1}{\beta}\left[t\ln\left(1 + \frac{a_0\beta}{\alpha}\right) - \frac{1}{\alpha}P_2(\xi)\right], \tag{10}$$

where $\xi = \frac{a_0\beta\exp(\beta t)}{(\alpha + a_0\beta)}$ and $P_2(\xi)$ is the polylogarithmic function

$$P_2(\xi) = \sum_{k=1}^{\infty}\frac{\xi^k}{k^2}. \tag{11}$$

Eq. (10) allows to calculate the analytical value of the loop area, which is a very important indicator of the level of hysteresis of the system, and consequently also of its non-linearity.

As a second application of the PUN method, we wish to investigate the time dependence of variables, such as the resonance frequency and the Q-factor of a consolidated granular sample, subjected to various protocols of varying temperature. In fact, in samples, subjected to thermal shock, a drop in the elastic modulus and an increase in the material damping have been observed [22,23]. After the shock is removed, the material properties recover toward their original values, but this process may take hours or days (hence the name: "Slow-Dynamics" [22]). An important feature of this effect is that the elastic modulus (and consequently the resonance frequency) decreases in response to the temperature change independently of the sign of the shock.

In the following we reports the results of a PUN analysis of the experimental data produced by T.J. Ulrich and collaborators at the Los Alamos National Laboratory [24]. An environmental control chamber was constructed to control the

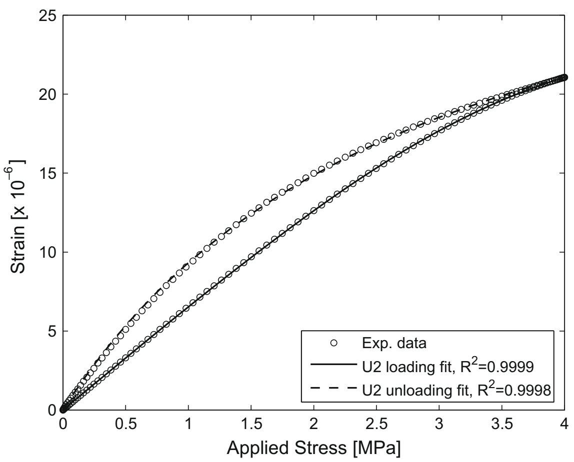

Fig. 1. Strain vs. stress in a typical elastodynamic hysteretic loop. The agreement between the U2 predictions (solid and dashed lines) and the results of virtual experiments [19] is excellent.

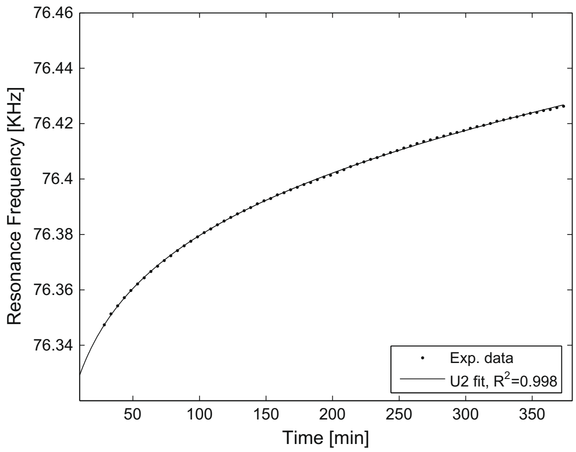

Fig. 2. Resonance frequency recovery vs. time after a thermal shock in a thermo-elastodynamic experiment on a Barea sandstone sample (from [26]).

temperature to within 0.01 K in a 760 torr dry He atmosphere. The He was introduced and the temperature was increased up to the value $T_0 = 330$ K and held there for 7 days to minimize the effect of thermally induced slow-dynamics at the initial temperature for the experiments to follow. Then the temperature was suddenly dropped down to the value $T_1 = 320$ K. The elastic state was monitored by measuring the resonance frequencies and Q-factor using Resonant Ultrasound Spectroscopy [25].

In Fig. 2 we show (for brevity) the result of only one of several curves, representing the recovery of the resonance frequency in correspondence with each sudden change in temperature. The fit of the experimental data by means of Eq. (6) is, as in all other cases, excellent ($R^2 = 0.998$). This suggests that Eq. (6) should be used to describe the recovery, rather than other equations, which have been proposed [26]. It is also remarkable that, although all the recovery experimental curves are similar and monotonically increasing with our formalism it is possible to discriminate between the curves following a downwards or upwards shock, by looking at the sign of the fitting parameter α in Eq. (6).

4. Conclusions

Scaling invariance is an extremely useful property of many systems of relevance in different disciplines [27]. It is, however, usually lost due to the non-linearity, which is an almost universal (albeit often negligible or neglected) consequence of the interaction of any system with its environment (or even seeded in the system itself).

In all phenomenologies belonging to a PUN class of special relevance ($U2$), the scaling invariance may, however, be retrieved by using suitable fractal-dimensioned variables. In addition to the obvious practical benefits, the procedure may applied to discriminate between successive phases in the evolution of a system. In fact, if the nature of the system changes, also the scaling invariance breaks down. Work in this direction, i.e. to pin-point the time of transition in material specimens for non-destructive evaluation purposes, is in progress.

The applicability and validity of the proposed approach are demonstrated by two instances of applications to two different problems in the field of the elasto-dynamics, which also illustrate the usefulness of the analysis.

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References

[1] West GB, Brown JH. The fourth dimension of life: fractal geometry and allometric scaling of organisms. Phys Today 2004;57:36.
[2] West GB, Brown JH, Enquist BJ. Science 1997;276:122.
[3] Enquist BJ, Niklas KJ. Nature 2001;276:410.
[4] West GB, Brown JH, Enquist BJ. Nature 2001;413:628.
[5] Gillooly JF et al. Nature 2002;417:70.
[6] Guiot C, Degiorgis O, Delsanto PP, Gabriele P, Deisboeck TS. J Theor Biol 2003;225:147.
[7] Delsanto PP, Guiot C, Degiorgis O, Condat CA, Mansury Y, Deisboeck TS. Appl Phys Lett 2004;85:2445.
[8] Delsanto PP, Griffa M, Condat CA, Delsanto S, Morra L. Phys Rev Lett 2005;94:148105.
[9] Castorina P, Delsanto PP, Guiot C. Phys Rev Lett 2006;96:188701.
[10] Delsanto PP, editor. Universality of non-classical non-linearity with applications to NDE and Ultrasonics. New York: Springer; 2007.
[11] Gompertz B. R Soc Lond 1825;123:513.
[12] Vladar de H. J Theor Biol 2006;238:245.
[13] Carpinteri A. Mech Mater 1994;18:89.

[14] Carpinteri A. Int J Solids Struct 1994;31:291.
[15] Kleiber M. The fire of life: an introduction to animal energetics. In: Robert E. Krieger, editor. Huntington:New York; 1975.
[16] Guiot C, Delsanto PP, Carpinteri A, Pugno N, Mansury Y, Deisboeck TS. J Theor Biol 2006;240:459.
[17] Delsanto PP, Guiot C, Gliozzi AS. Theoret Biol Med Model 2008;5:5.
[18] Guyer RA, Johnson PA. Phys Today 1999;52:30.
[19] Gliozzi AS. unpublished results.
[20] Delsanto PP, Scalerandi M. Phys Rev B 2003;68:064107.
[21] Scalerandi M, Nobili M, Griffa M, Gliozzi AS, Bosia F. Phys Rev B 2006;73:092103.
[22] Guyer R, Johnson PA. J Mater Process Manuf Sci 2000;9:14.
[23] TenCate JA, Smith E, Guyer RA. Phys Rev Lett 2000;85:1020.
[24] Ulrich TJ. AIP Seventeenth international symposium on nonlinear acoustics 2005; 2006;27:838.
[25] Migliori A, Sarrao JL, editors. Resonant ultrasound spectroscopy. New York: Wiley; 1996.
[26] Delsanto PP, Gliozzi AS, Ulrich TJ. Elastic response to temperature variations in granular media: a phenomenological analysis, Appl Phys Lett 2008, submitted for publication.
[27] Carpinteri A, Pugno N. Nat Mat 2005;4:421.