A Proposition for a "Self-Consistent" Gradient Elasticity

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ABSTRACT

The notion of "self-consistent" boundary conditions in gradient elasticity is explored. They are introduced in the place of the "standard" boundary conditions commonly used in the formulation of gradient elasticity problems derived through corresponding variational principles. The case of a perforated membrane under biaxial tension is solved, as an example. The predicted hole size-effect is then compared with the solutions of classical and gradient elasticity and with that obtained by a "quantized elasticity" approach. Only self-consistent gradient elasticity and the quantized approach seem to provide, in a convenient way, fully realistic results in the asymptotic regime.

1. INTRODUCTION

Generalized theories of linear elasticity involving higher-order strain gradients have been revived recently starting with the early work of Aifantis and co-workers /1-5/ which continues up to the present time /6-9/ with significant contributions by many researchers including Vardoulakis et al /10/, Exadaktylos et al /11/, Polizotto et al /12/, Aravas et al /13/, Beskos et al /14/, Georgiadis et al /15/, Giannakopoulos et al /16/, and others (e.g. /17-21/).

All of the above works are essentially based on the simple model of gradient elasticity advocated by Aifantis /1/ involving only one extra constant, commonly known as gradient coefficient, the square root of which may be physically identified with the dominant internal length defining the extent of nonlocality in the material system under consideration. This model, which was also used to interpret size effects in torsion and bending of elastic materials with microstructure and compare them with predictions of Cosserat elasticity /22/, could be directly obtained from a nonlinear gradient elasticity theory advocated by Triantafyllidis and Aifantis /23/ through a direct analogy to the gradient plasticity theory previously introduced by Aifantis /24-25/. This theory is based on a correction of the strain energy function by one gradient term only in analogy to van der Waals thermodynamic theory of liquid-vapor transition, as discussed in the mechanical theory of fluid interfaces of Aifantis and Serrin /26/. It thus enjoys a different physical motivation than, for example, Toupin's /27/ and Mindlin's /28/ celebrated works on generalized elasticity theories which involve many constants and were mainly applied to wave propagation studies. In this connection, it is pointed out that several of the above gradient elasticity papers refer only to Mindlin's works without citing Aifantis' model which is exactly what they eventually use in their analyses (see, for example, the works by Georgiadis et al /15/). A slightly more general model including both stress and strain gradients was outlined by Aifantis /29/ in a review on applications of gradient theory to "ill-posed" problems of elasticity, plasticity and dislocation dynamics, with emphasis, respectively, on eliminating elastic singularities from dislocation lines and crack tips, on obtaining shear band widths and spacings in plasticity on the micron scale along with a corresponding interpretation of size effects and, finally, on interpreting dislocation patterning phenomena at the mesoscale.

Various types of boundary conditions have been used in the aforementioned works to solve corresponding boundary value problems. They involve the usual boundary conditions of classical elasticity, as well as additional boundary conditions required as a result of the introduction of gradient terms. These extra boundary conditions are usually obtained in connection with the well-posedeness and uniqueness of related boundary value problems or through appropriate variational principles. Their physical meaning and experimental realization is usually difficult to implement. Thus, a different procedure is explored here by associating the necessary extra boundary conditions with the specific problem at hand and choosing them in a "self-consistent" manner, in accordance with a more physical perspective.

The corresponding "self-consistent" boundary conditions are able to remove the paradoxes associated with classical elasticity, that may only partially be removed if standard "extra boundary conditions" are used. An example is provided in this paper, where the elastic problem is solved within a self-consistent gradient elasticity framework, for a perforated membrane under biaxial tension. The hole size-effect is then compared with the solutions of classical elasticity, gradient elasticity with non self-consistent extra boundary conditions, and with that obtained by Novozhilov's approach /30/, that is the stress-analogue of the energy based quantized fracture mechanics /31/. Only the self-consistent gradient elasticity and quantized approaches /30-31/ seem to provide fully realistic asymptotic matching results.

2. THEORY

The simple version of gradient elasticity theory to be used here is of the form

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\left(\sigma_{kk}\right)\delta_{ij} - c\frac{\nu}{E}\left(\nabla^2\sigma_{kk}\right)\delta_{ij} \ , \tag{1}$$

where $(\varepsilon_{ij}, \sigma_{ij})$ denote the stress and strain tensors, (ν, E) are the usual elastic moduli and c is the gradient coefficient having dimensions of length square ($c \equiv \ell^2$; ℓ is an internal length associated with the underlying microstructure of the gradient elastic medium). This simplified model was used in /32/ and it is a special case of the gradient elasticity model used in /29/. It suggests that hydrostatic pressure gradients are directly influencing the stress-strain relation and a simple physical basis for it may be obtained as follows.

Let us start with a standard elastic medium for which the strain is determined by the stress as in Hooke's law and also, in addition, by a scalar internal variable φ, representing a microscopic porosity/void variable or another degree of freedom. Then we may write

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + k\phi\delta_{ij} , \tag{2}$$

where k is a constant. The internal variable is assumed to obey a "complete balance law" containing both a rate and a flux term /33/, i.e.

$$\dot{\phi} + \mathrm{div}\mathbf{j} = g , \tag{3}$$

where **j** is the flux of the internal variable within the elementary volume and g its production. In a simple linear theory, the flux **j** may be taken to be proportional to the gradient $\nabla\phi$ of the internal variable, while the source term g may be taken as a linear function of the hydrostatic stress σ and the internal variable itself, i.e.

$$\mathbf{j} = -D\nabla\phi \qquad \text{and} \qquad g = -\Lambda\sigma - M\phi , \tag{4}$$

where (D, Λ, M) are positive constants. The plus sign in the last term of Eq. (2) indicates that extra strain is produced as a result of the action of φ, while the minus signs in Eq. (4) indicate that, in the case where the microstructure is of the form of void space, damage "migrates" from "weak" to "strong" regions, while "healing" takes place under the action of tensile stress and damage evolution proceeds in a stable manner. On combining Eqs. (3) and (4) and taking the Fourier transform, we have

$$\dot{\phi}_q = -Dq^2\phi_q - \Lambda\sigma_q - M\phi_q \,, \tag{5}$$

where the subscript q denotes the Fourier transform of the respective variable where q is the corresponding magnitude of the wave vector. By assuming that ϕ_q varies rapidly in comparison to the measured stress and strain (i.e. the lifetime of structured defects represented by the variable φ is much smaller than the corresponding time scales over which macroscopic variables evolve), the adiabatic elimination argument ($\dot{\phi}_q \approx 0$; see, for example, /34/) leads to the relation

$$\varphi_q = -\frac{\Lambda}{M + Dq^2}\sigma_q \,, \tag{6}$$

which, by adopting a Taylor's series expansion for the term $\Lambda/\left(M + Dq^2\right)$ on the assumption that $\left(Dq^2/M\right) << 1$, gives,

$$\varphi_q = -\frac{\Lambda}{M}\sigma_q - \frac{\Lambda D}{M^2}q^2\sigma_q \;\Rightarrow\; \varphi = -\frac{\Lambda}{M}\sigma - \frac{\Lambda D}{M^2}\nabla^2\sigma \;, \tag{7}$$

where the hydrostatic stress variable σ may be replaced with the trace of the stress tensor σ_{kk}. Then, substitution of Eq. (7) into Eq. (2) yields

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \left(\frac{\nu}{E} + \frac{k\Lambda}{M}\right)(\sigma_{kk})\,\delta_{ij} - \frac{k\Lambda D}{M^2}\left(\nabla^2\sigma_{kk}\right)\delta_{ij} \;, \tag{8}$$

which on setting $c = \left(k\Lambda D/M^2\right)\left(E/\nu\right)$ and assuming the factor $\left(k\Lambda/M\right)$ can be neglected with respect to $\left(\nu/E\right)$, or that $\left(cM/D\right)$ can be neglected with respect to unity, Eq. (1) can be obtained. [This assumption could be lifted by considering more general evolution for the internal variable φ, for example, by allowing a stress gradient term to enter in Eq. $(4)_1$.]

It should be pointed out that the above microscopic substantiation of the gradient-dependent elastic constitutive law given by Eq. (1), provides only one possible justification for the proposed modification of Hooke's law by the Laplacian $\nabla^2\sigma_{kk}$ of the hydrostatic stress. Other types of mechanisms may be invoked to obtain other types of gradient dependence as discussed by the last author in /34/ (see also /29/). Within a more rigorous derivation, atomistic and molecular dynamics arguments may be used to substantiate the constitutive assumptions embodied in Eqs. (2) and (4). On the other hand, such type of MD simulations may be used directly in conjunction with Eq. (1), independently of the underlying physical mechanism leading to the extra Laplacian term $\nabla^2\sigma_{kk}$, in order to provide the needed information on the gradient coefficient c.

3. PERFORATED MEMBRANE IN BIAXIAL TENSION

Consider the case of an infinitely large membrane containing a hole of radius α, under biaxial remote load σ, for which the following gradient constitutive equation holds

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\left(\sigma_{kk}\right)\delta_{ij} - c\frac{\nu}{E}\left(\nabla^2\sigma_{kk}\right)\delta_{ij} \ , \tag{9}$$

where $(\varepsilon_{ij}, \sigma_{ij})$ are the stress and strain tensors, ν and E are the Poisson ratio and Young modulus, δ_{ij} is the Kronecker delta and c is the gradient coefficient. Consider plane stress and polar coordinates. Combining the constitutive law with the compatibility and equilibrium equations allows us to solve the problem for a constitutive law given by Eq. (9) in the form /32/

$$\sigma_r = C_1\left(1+2\ln r\right) + 2C_2 + \frac{C_3}{r^2} + \frac{1}{r\sqrt{c'}}\left[D_1 I_1\left(\frac{r}{\sqrt{c'}}\right) - D_2 K_1\left(\frac{r}{\sqrt{c'}}\right)\right] , \tag{10}$$

$$\begin{aligned}\sigma_\theta &= C_1\left(3+2\ln r\right) + 2C_2 - \frac{C_3}{r^2} + \\ &+ \frac{1}{\sqrt{c'}}\left[\frac{D_1}{\sqrt{c'}} I_0\left(\frac{r}{\sqrt{c'}}\right) - \frac{D_1}{r} I_1\left(\frac{r}{\sqrt{c'}}\right) + \frac{D_2}{\sqrt{c'}} K_0\left(\frac{r}{\sqrt{c'}}\right) + \frac{D_2}{r} K_1\left(\frac{r}{\sqrt{c'}}\right)\right],\end{aligned} \tag{11}$$

where $c' = c\nu$, C_i, D_i are constants and I_n, K_n are the modified Bessel functions of first and second kind respectively. In order to have limited stresses for $r = \alpha \to 0$, the constant C_1 must vanish. The other four constants C_2, C_3, D_1, D_2 should be derived according to the relevant boundary conditions. Before we proceed with their determination we outline first the derivation of the general solution for the stresses given by Eqs. (10) and (11), and the corresponding expressions for the strains.

The procedure for obtaining this solution is detailed in /32/ and is also summarized here. A stress function Φ is introduced such that in polar coordinates (r, θ) we have

$$\sigma_r = \frac{1}{r}\frac{d\Phi}{dr}, \qquad \sigma_\theta = \frac{d^2\Phi}{dr^2} \ , \tag{12}$$

while the corresponding strains are given by

$$\begin{aligned}\varepsilon_r &= \frac{1}{E}(\sigma_r - \nu\sigma_\theta) - c\frac{\nu}{E}\nabla^2(\sigma_r + \sigma_\theta) \ , \\ \varepsilon_\theta &= \frac{1}{E}(-\nu\sigma_r + \sigma_\theta) - c\frac{\nu}{E}\nabla^2(\sigma_r + \sigma_\theta) \ ,\end{aligned} \tag{13}$$

which by using the compatibility equation

$$\frac{d^2\varepsilon_\theta}{dr^2}+\frac{2}{r}\frac{d\varepsilon_\theta}{dr}-\frac{1}{r}\frac{d\varepsilon_r}{dr}=0\,, \tag{14}$$

leads to the following sixth-order differential equation for $\Phi(r)$

$$\nabla^4(1-c\nu\nabla^2)\,\Phi=0\,. \tag{15}$$

By setting

$$(1-c'\nabla^2)\,\Phi=\Phi^E\,;\qquad c'=c\nu\,,\qquad (c'>0) \tag{16}$$

Eq. (15) becomes

$$\nabla^4\Phi^E=0\,;\qquad \nabla^4\Phi^E=\nabla^2(\nabla^2\Phi^E)=\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\,\Phi^E}{dr^2}+\frac{1}{r}\frac{d\,\Phi^E}{dr}\right), \tag{17}$$

the solution of which for axial symmetric problems has the familiar from linear elasticity form

$$\Phi^E=C_1r^2\ln r+C_2r^2+C_3\ln r+C_4\,. \tag{18}$$

By inserting Eq. (18) into Eq. (16) we have

$$\frac{d^2\Phi}{dx^2}+\frac{1}{x}\frac{d\Phi}{dx}-\Phi=-(C_1r^2\ln r+C_2r^2+C_3\ln r+C_4)\,, \tag{19}$$

where $x=r/\sqrt{c'}$. This is a standard differential equation of Bessel type with solution /35, 32/

$$\Phi=D_1I_0\left(\frac{r}{\sqrt{c'}}\right)+D_2K_0\left(\frac{r}{\sqrt{c'}}\right)+(C_1r^2+C_3)\ln r+(C_2r^2+C_4)\,, \tag{20}$$

where (D_1, D_2, C_1, C_2, C_3, C_4) are constants and (I_0, K_0) are modified Bessel functions of zero order of the first and second kind, respectively. The $C_1=0$ for the circular hole problem in order that the tangential displacement to be single-valued (at $\theta=0$ and $\theta=2\pi$). It follows that the appropriate expressions for the stresses and strains read /32/

$$\sigma_r = \frac{1}{r\sqrt{c'}}\left[D_1 I_1\left(\frac{r}{\sqrt{c'}}\right) - D_2 K_1\left(\frac{r}{\sqrt{c'}}\right)\right] + \frac{C_3}{r^2} + 2C_2 \ ,$$

$$\sigma_\theta = \frac{1}{\sqrt{c'}}\left[\begin{array}{l}\frac{D_1}{\sqrt{c'}} I_0\left(\frac{r}{\sqrt{c'}}\right) - \frac{D_1}{r} I_1\left(\frac{r}{\sqrt{c'}}\right) \\ + \frac{D_2}{\sqrt{c'}} K_0\left(\frac{r}{\sqrt{c'}}\right) + \frac{D_2}{r} K_1\left(\frac{r}{\sqrt{c'}}\right)\end{array}\right] - \frac{C_3}{r^2} + 2C_2 \ , \tag{21}$$

and

$$\varepsilon_r = \frac{1+\nu}{E\sqrt{c'}}\left[\begin{array}{l}\frac{D_1}{r} I_1\left(\frac{r}{\sqrt{c'}}\right) - \frac{D_1}{\sqrt{c'}} I_0\left(\frac{r}{\sqrt{c'}}\right) \\ - \frac{D_2}{r} K_1\left(\frac{r}{\sqrt{c'}}\right) - \frac{D_2}{\sqrt{c'}} K_0\left(\frac{r}{\sqrt{c'}}\right)\end{array}\right] + \frac{1+\nu}{E}\left[\frac{C_3}{r^2} + \frac{2(1-\nu)}{1+\nu} C_2\right] \ ,$$

$$\varepsilon_\theta = \frac{-(1+\nu)}{E}\left\{\frac{1}{\sqrt{c'}}\left[\frac{D_1}{r} I_1\left(\frac{r}{\sqrt{c'}}\right) - \frac{D_2}{r} K_1\left(\frac{r}{\sqrt{c'}}\right)\right] + \frac{C_3}{r^2} - \frac{2(1-\nu)}{1+\nu} C_2\right\} \ . \tag{22}$$

On returning to the determination of the constants (D_1, D_2, C_2, C_3), we first use the standard boundary conditions

$$\sigma_r = 0 \quad \text{for} \quad r = \alpha \ ; \quad \sigma_r = \sigma \quad \text{for} \quad r \to \infty \ , \tag{23}$$

and the extra boundary conditions used in previous works on gradient elasticity (e.g. /2-3/), i.e.

$$d^2u / dr^2 = 0 \quad \text{for} \quad r = \alpha \quad \text{and} \quad r \to \infty \ , \tag{24}$$

where u denotes the radial component of the displacement field. From Eq. $(22)_1$ we have

$$\frac{d^2u}{dr^2} = \frac{d\varepsilon_r}{dr} = \frac{1+\nu}{E}\left\{\frac{D_1}{c'}\left[\frac{I_2\left(\frac{r}{\sqrt{c'}}\right)}{r} - \frac{I_1\left(\frac{r}{\sqrt{c'}}\right)}{\sqrt{c'}}\right] + \frac{D_2}{c'}\left[\frac{K_2\left(\frac{r}{\sqrt{c'}}\right)}{r} + \frac{K_1\left(\frac{r}{\sqrt{c'}}\right)}{\sqrt{c'}}\right] - \frac{2C_3}{r^3}\right\}, \tag{25}$$

and, then, Eqs. $(21)_1$ and (25) can be combined with Eqs. (23) and (24) to give

$$D_1 = 0 \ , \quad D_2 = \frac{-2\sigma c'}{T_h'} \ , \quad C_2 = \frac{\sigma}{2} \ , \quad C_3 = -\alpha\sigma\left[\alpha + \frac{2\sqrt{c'}K_1(h')}{T_h'}\right], \tag{26}$$

where the dimensionless quantities h' and T_h' are defined by the relation $h' = \alpha/\sqrt{c'}$ and $T_h' = h'K_1(h') + K_0(h')$. The final expressions for the stresses σ_r and σ_θ are given by

$$\sigma_r = \sigma\left(1 - \frac{\alpha^2}{r^2}\right) - \frac{2\sigma\sqrt{c'}}{r\ T_h'}F'(r), \qquad \sigma_\theta = \sigma\left(1 + \frac{\alpha^2}{r^2}\right) + \frac{2\sigma}{T_h'}\left\{\frac{\sqrt{c'}}{r}F'(r) - K_0\left(\frac{r}{\sqrt{c'}}\right)\right\}, \tag{27}$$

where $r \geq \alpha$ and $F'(r) = (\alpha/r)K_1(h') - K_1(r/\sqrt{c'})$.

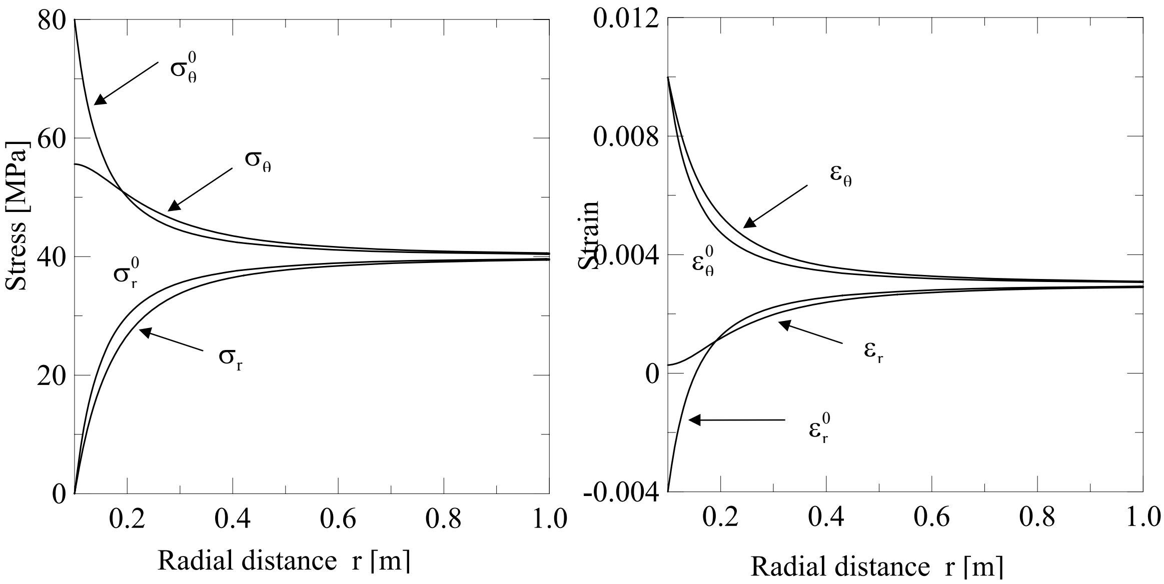

Fig. 1: Plots of (i) radial and tangential stresses, and (ii) radial and tangential strains, in classical (dotted lines) and gradient (solid lines) elasticity for α=0.1m, σ=40 MPa, ν=0.4, E=8 GPa, $\sqrt{c} = 86$ mm, h'=1.826.

The corresponding expressions for the strains (ε_r, ε_θ) and the displacement (u) are given by

$$\varepsilon_r = \frac{\sigma(1+\nu)}{E}\left\{\frac{1-\nu}{1+\nu} - \frac{\alpha^2}{r^2} - \frac{2\sqrt{c'}}{T_h'}\left[\frac{1}{r}F'(r) - \frac{1}{\sqrt{c'}}K_0\left(\frac{r}{\sqrt{c'}}\right)\right]\right\},$$
$$\varepsilon_\theta = \frac{\sigma(1+\nu)}{E}\left\{\frac{1-\nu}{1+\nu} + \frac{\alpha^2}{r^2} + \frac{2\sqrt{c'}}{r\ T_h'}F'(r)\right\}, \tag{28}$$

and

$$u = \frac{\sigma(1+\nu)}{E}\left\{\frac{1-\nu}{1+\nu}r + \frac{\alpha^2}{r} + \frac{2\sqrt{c'}}{T_h'}F'(r)\right\}. \tag{29}$$

The plots of Eqs. (27) and (28) are given in Figure 1 for a set of arbitrarily chosen values of the material parameters. It is seen from these plots that significant differences arise only in the neighborhood of the hole. This motivates a more careful consideration of the stress concentration factor. In fact, the tangential stress at the boundary of the hole is calculated form Eq. (27) as

$$\sigma_\theta\big|_{r=a} = 2\sigma - \frac{2\sigma}{T_h'} K_0(h') . \qquad (30)$$

By assuming further that failure occurs when the tangential stress at the hole boundary attains a critical value (maximum stress failure criterion of Rankine type), one may derive the following expressions for the stress intensity factor $S_c\ (= \sigma_\theta/\sigma)_{r=0}$ and the dimensionless failure stress $\sigma_f\ (= \sigma/\sigma_\theta)_{r=a,\,\mathrm{failure}}$

$$S_c = 2\left(1 - \frac{K_0(h')}{h'K_1(h') + K_0(h')}\right), \qquad \sigma_f' = \frac{1}{2}\left(1 + \frac{K_0(h')}{h'K_1(h')}\right) , \qquad (31)$$

where it was assumed the at failure σ_θ at $r = \alpha$ attains a critical value σ^* which is a material constant. The corresponding plots are given in Figure 2.

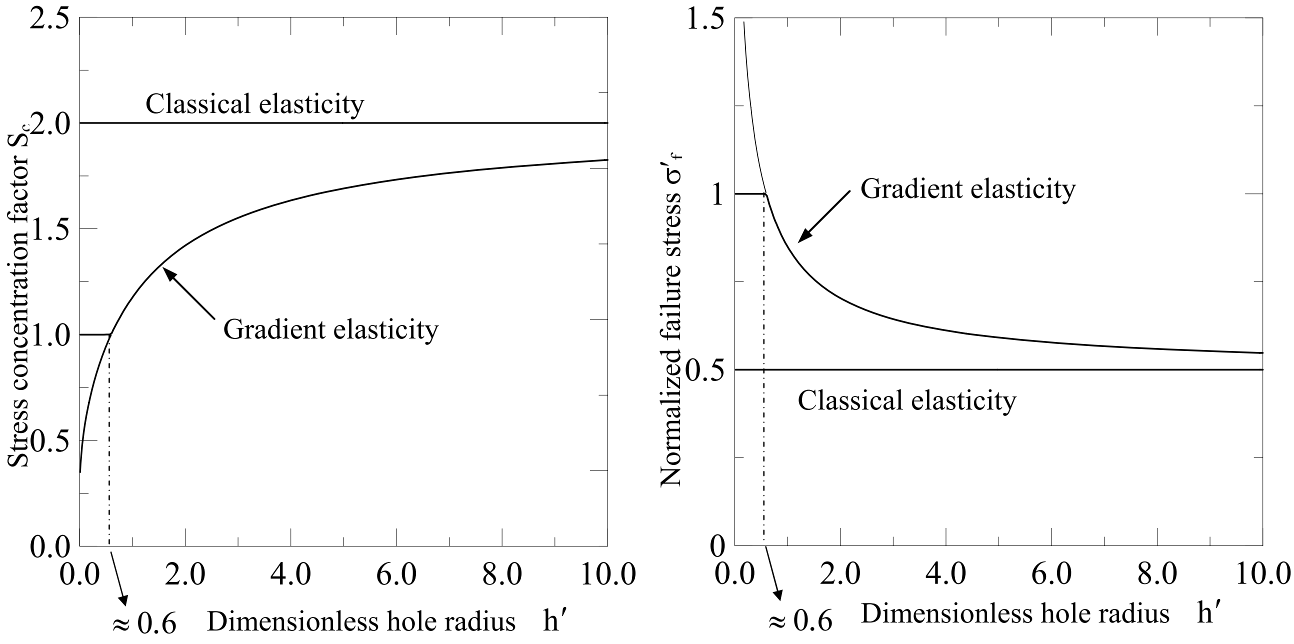

Fig. 2: Variation of (i) the stress concentration factor S_c, and (ii) the normalized failure stress σ_f' as a function of the dimensionless hole radius h'.

4. SELF-CONSISTENT BOUNDARY CONDITIONS

In this final section we introduce the notion of self-consistent boundary conditions in the following sense. Instead of adopting the displacement extra boundary conditions employed earlier, i.e.

$$\frac{\partial^2 u}{\partial r^2}(r=\alpha)=0\,,\qquad \frac{\partial^2 u}{\partial r^2}(r\to\infty)=0\quad, \tag{32}$$

by requiring the solution to obey the following constraint

$$\sigma_\theta\left(\alpha\to 0\right)=\sigma\quad, \tag{33}$$

from which we derive $D_2\left(\alpha\to 0\right)=\sigma c'\big/\ln\left(\alpha/\sqrt{c'}\right)$ and since we need a limited value for $D_2\left(\alpha\to\sqrt{c'}\right)$ we fix $D_2=\sigma c'\left(1-\alpha/\sqrt{c'}\right)\big/\ln\left(\alpha/\sqrt{c'}\right)$, introducing the screening function (of the natural logarithm) $1-\alpha\big/\sqrt{c'}$ (tending to zero for $\alpha\to 0$ as required). As for C_1, $D_1=0$ in order to have the stress limited. From $\sigma_r\left(r\to\infty\right)=\sigma$, $C_2=\sigma/2$, whereas from $\sigma_r\left(r=\alpha\right)=0$, $C_3=-\alpha^2\sigma+\alpha/\sqrt{c'}\,D_2K_1\left(\alpha/\sqrt{c'}\right)$. Thus, a new "self-consistent" solution may thus be obtained. In passing, we remark that the above "self-consistent" method should be viewed only as an alternative in deducing the appropriate form of the extra boundary conditions. From a "traditional mechanics" point of view the extra boundary conditions are obtained from variational principles and this approach has let to complex boundary conditions (see, for example, Mindlin /28/, Tsagrakis /36/, Aifantis and Askes /21/), the physical meaning of which and its experimental realization may be difficult to implement. It is thus left up to the experiment and the particular problem at hand to suggest the most convenient and physically meaningful way to introduce the appropriate form of the extra boundary conditions. For example, looking at the stress concentration factor near the hole, defined by $S_c=\sigma_\theta\left(r=\alpha\right)\big/\sigma$, we derive for the above self-consistent approach, in contrast to classical elasticity for which

$$S_c^E=2\quad, \tag{34}$$

and the gradient elasticity with the standard extra boundary conditions given by Eq. (32) for which S_c^{GE} is given by Eq. (31), the following expression:

$$S_c^{GE/SC}=2+\frac{K_0\left(\alpha/\sqrt{c'}\right)\left(1-\alpha/\sqrt{c'}\right)}{\ln\left(\alpha/\sqrt{c'}\right)}\,. \tag{35}$$

Note that $S_c^{GE/SC}(\alpha \to \infty) = 2$ as in classical elasticity, whereas $S_c^{GE/SC}(\alpha \to 0) = 1$ as required by the self-consistent boundary condition.

By using non self-consistent boundary conditions, i.e. gradient elasticity with the more standard boundary conditions given by Eq. (32), we have the result of Eq. $(31)_1$ which can be re-written as

$$S_c^{GE} = 2\left(1 - \frac{K_0\left(\alpha/\sqrt{c'}\right)}{\alpha/\sqrt{c'}\,K_1\left(\alpha/\sqrt{c'}\right) + K_0\left(\alpha/\sqrt{c'}\right)}\right). \tag{36}$$

In this case, it is noted that $S_c^{GE}(\alpha \to 0) = 0$. This result may be considered as not acceptable, as it suggests a defect-free membrane of infinite strength; even though at the other limit at infinity, the behavior is as in classical elasticity, i.e. $S_c^{GE}(\alpha \to \infty) = 2$. In concluding, we remark that by applying the quantized approach /30-31/ we derive the following stress concentration factor

$$S_c^{QFM} = \frac{2 + a/\alpha}{1 + a/\alpha} \quad , \tag{37}$$

where a is the fracture quantum. Note that $S_c^{QFM}(\alpha \to 0) = 1$ and $S_c^{QFM}(\alpha \to \infty) = 2$, i.e. the same realistic limits as for $S_c^{GE/SC}$ are obtained. Furthermore, we note that since $S_c^{GE/SC}(\alpha \to 0) = 1 + \alpha/\sqrt{c'}$ and $S_c^{GE/SC}(\alpha \to 0) = 1 + \alpha/a$, it is evident that the connection between the two theories is established by the relation $a \approx \sqrt{c'}$, a quite interesting result suggesting that the fracture quantum equals to the internal length. By considering the dimensionless hole size $\alpha^* = \alpha/a = \alpha/\sqrt{c'} = h'$, the four different solutions of Eqs. (34), (35), (36) and (37) are compared in Figure 3. Thus, most reasonable solutions for the hole-size effect are provided by the predictions of Eqs. (35) and (37).

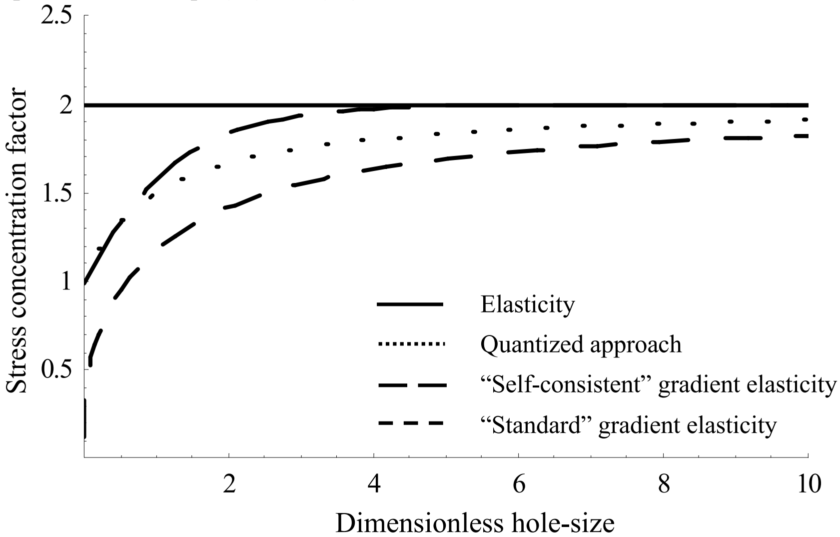

Fig. 3: Comparison between predicted hole size-effects for various elasticity-based theories.

5. CONCLUSIONS

The main thrust of this paper was a proposal for the formulation of a self-consistent gradient elasticity. Standard "extra boundary conditions" are substituted by "self-consistent" boundary conditions. The case of a perforated membrane under biaxial tension is treated as an example, but the proposed modification has general validity.

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REFERENCES

1. E.C. Aifantis, On the role of gradients in the localization of deformation and fracture, *Int. J. Engrg. Sci.* **30**, 1279-1299 (1992).
2. B. Altan and E.C. Aifantis, On the structure of the mode III crack-tip in gradient elasticity, *Scripta Met. Mater.* **26**, 319-324 (1992).
3. C.Q. Ru and E.C. Aifantis, A simple approach to solve boundary value problems in gradient elasticity, *Acta Mechanica* **101**, 59-68 (1993).
4. D.J. Unger and E.C. Aifantis, The asymptotic solution of gradient elasticity for mode III, *Int. J. Fracture* **71**, R27-R32 (1995). [see also: D.J. Unger and E.C. Aifantis, Strain gradient elasticity theory for antiplane shear cracks. Part I: Oscillatory displacements, *Theor. Appl. Fract. Mech.* **34**, 243-252 (2000); D.J. Unger and E.C. Aifantis, Strain gradient elasticity theory for antiplane shear cracks. Part II: Monotonic displacements, *Theor. Appl. Fract. Mech.* **34**, 253-265 (2000)].
5. M.Yu. Gutkin and E.C. Aifantis, Screw dislocation in gradient elasticity, *Scripta Materialia* **35**, 1353-1358 (1996). [see also: M.Yu. Gutkin and E.C. Aifantis, Edge dislocation in gradient elasticity, *Scripta Mater.* **36**, 129-135 (1997); M.Yu. Gutkin and E.C. Aifantis, Dislocations and disclinations in gradient elasticity, *Phys. Status Solidi* **214B**, 245-284 (1999); M.Yu. Gutkin and E.C. Aifantis, Dislocations in the theory of gradient elasticity, *Scripta Mater.* **40**, 559-566 (1999)].
6. H. Askes and E.C. Aifantis, Gradient elasticity theories in statics and dynamics – a unification of approaches, *Int. J. Fracture* **139**, 297-304 (2006). [see also: H. Askes, T. Bennett and E.C. Aifantis, A new formulation and C^0-implementation of dynamically consistent gradient elasticity, *Int. J. Numer.*

Meth. Engng. **72**, 111-126 (2007)].

7. M. Lazar, G.A. Maugin and E.C. Aifantis, On dislocations in a special class of generalized elasticity, Phys. Status Solidi B 242, 2365-2390 (2005). [see also: M. Lazar, G.A. Maugin and E.C. Aifantis, On a theory of nonlocal elasticity of bi-Helmholtz type and some applications, *Int. J. Solids Struct.* **43**, 1404-1421 (2006); M. Lazar, G.A. Maugin and E.C. Aifantis, Dislocations in second strain gradient elasticity, *Int. J. Solids Struct.* **43**, 1787-1817 (2006)].
8. G. Rambert, J.-C. Grandidier and E.C. Aifantis, On the direct interactions between heat transfer, mass transport and chemical processes within gradient elasticity, *Eur. J. Mechan. A. Solids* **26**, 68-87 (2007).
9. J. Kioseoglou, G.P. Dimitrakopulos, Ph. Komninou, Th. Karakostas, I. Konstantopoulos, M. Avlonitis and E.C. Aifantis, Analysis of partial dislocations in wurtzite GaN using gradient elasticity, *Phys. Status Solidi* **A203**, 2161 – 2166 (2006).
10. I. Vardoulakis, Linear Micro-elasticity, in: *Degradations and Instabilities in Geomaterials*, CISM Lectures No. 461, eds. F. Darve and I. Vardoulakis, Springer-Verlag, Wien-New York, pp. 107-149 (2004) [see also: I. Vardoulakis, G. Exadaktylos and E.C. Aifantis, Gradient elasticity with surface energy: Mode-III crack problem, *Int. J. Solids Struct.* **33**, 4531-4559 (1996). G. Exadaktylos, I. Vardoulakis and E.C. Aifantis, Cracks in gradient elastic bodies with surface energy, *Int. J. Fracture* **79**, 107-119 (1996); I. Vardoulakis and H.G. Georgiadis, SH surface waves in a homogeneous gradient-elastic half-space with surface energy, *J. Elasticity* **47**, 147-165 (1997)].
11. G.E. Exadaktylos and E.C. Aifantis, Two and three dimensional crack problems in gradient elasticity, *J. Mech. Behav. Mater.* **7**, 93-117 (1996). [see also: G.E. Exadaktylos, Gradient elasticity with surface energy: Mode-I crack problem, *Int. J. Solids Struct.* **35**, 421-456 (1998); G. Exadaktylos, Some basic half-plane problems of the cohesive elasticity theory with surface energy, *Acta Mech.* **133**, 175-198 (1999); G. Exadaktylos and I. Vardoulakis, Microstructure in linear elasticity and scale effects: a reconsideration of basic rock mechanics and rock fracture mechanics, *Tectonophysics* **335**, 81-109 (2001)].
12. C. Polizzotto, Unified thermodynamic framework for nonlocal/gradient continuum theories, *Eur. J. Mech. A-Solids* **22**, 651-668 (2003). [see also: C. Polizzotto, Nonlocal elasticity and related variational principles, *Int. J. Solids Struct.* **38**, 7359-7380 (2001)].
13. E. Amanatidou and N. Aravas, Mixed finite element formulations of strain-gradient elasticity, *Comp. Methods Appl. Mech. & Engng.* **191**, 1723-1751 (2002).
14. D. Polyzos, K.G. Tsepoura, S.V. Tsinopoulos and D.E. Beskos, A boundary element method for solving 2-D and 3-D static gradient elastic problems. Part I: Integral formulation, *Comput. Methods Appl. Mech. Engrg.* **192**, 2845–2873 (2003). [see also: K.G. Tsepoura, S.V. Tsinopoulos, D. Polyzos and D.E. Beskos, A boundary element method for solving 2-D and 3-D static gradient elastic problems. Part II: Numerical implementation, *Comput. Methods Appl. Mech. Engrg.* **192**, 2875-2907 (2003); D. Polyzos, K.G. Tsepoura and D.E. Beskos, Transient dynamic analysis of 3-D gradient elastic solids by BEM,

Computers and Structures **83**, 783–792 (2005)].

15. H.G. Georgiadis, The Mode-III crack problem in microstructured solids governed by dipolar gradient elasticity: Static and dynamic analysis, *J. Appl. Mech.-T. ASME* **70**, 517-530 (2003). [see also: H.G. Georgiadis and E.C. Aifantis, A gradient elasticity solution for the problem of anti-plane shear pulsating force on the surface of a half space, in: Proc. 4th Nat. Congress of Theor. Appl. Mech., Eds. P.S. Theocaris and E.E. Gdoutos, pp. 202-206, Xanthi, Greece (1995)].
16. A.E. Giannakopoulos, E. Amanatidou and N. Aravas, A reciprocity theorem in linear gradient elasticity and the corresponding Saint-Venant principle, *Int. J. Solids Struct.* **43** 3875–3894 (2006). [see also: A.E. Giannakopoulos, K. Stamoulis, Structural analysis of gradient elastic components, *Int. J. Solids Struct.* **44**, 3440–3451 (2007)].
17. A. Menzel and P. Steinmann, On the continuum formulation of higher gradient plasticity for single and polycrystals, *J. Mech. Phys. Solids* **48**, 1777-1796 (2000).
18. G.H. Paulino, A.C. Fannjiang and Y.S. Chan, Gradient elasticity theory for mode III fracture in functionally graded materials – Part I: Crack perpendicular to the material gradation, *J. Appl. Mech. – T. ASME* **70**, 531-542 (2003).
19. A.C. Fannjaiang, Y.S. Chan and G.H. Paulino, Strain gradient elasticity or antiplane shear cracks: A hypersingular integrodifferential equation approach, *SIAM J. Appl. Math.* **62**, 1066-1091 (2002).
20. V.K. Kalpakides and A.K. Agiasofitou, On material equations in second gradient electroelasticity, *J. Elasticity* **67**, 205-227 (2002).
21. K.E. Aifantis and H. Askes, Gradient elasticity with interfaces as surfaces of discontinuity for the strain gradient, *J. Mechan. Behav. Mater.* **18**, 283-306 (2007).
22. E.C. Aifantis, Strain gradient interpretation of size effects, *Int. J. Fracture* **95**, 299-314 (1999).
23. N. Triantafyllidis and E.C. Aifantis, A gradient approach to localization of deformation - I. Hyperelastic materials, *J. of Elasticity* **16**, 225-238 (1986).
24. E.C. Aifantis, On the microstructural origin of certain inelastic models, *Trans. ASME*, *J. Engng. Mat. Techn.* **106**, 326-330 (1984).
25. E.C. Aifantis, The Physics of plastic deformation, *Int. J. Plasticity* **3**, 211-247 (1987). [see also: E.C. Aifantis, On the role of gradients in the localization of deformation and fracture, *Int. J. Engrg. Sci.* **30**, 1279-1299 (1992); E.C. Aifantis, Pattern formation in plasticity, *Int. J. Eng. Sci.* **33**, 2161-2178 (1995)].
26. E.C. Aifantis and J.B. Serrin, The mechanical theory of fluid interfaces and Maxwell's rule, *J. of Colloid Inter. Sci.* **96**, 517-529 (1983). [see also: E.C. Aifantis and J.B. Serrin, Equilibrium solutions in the mechanical theory of fluid microstructures, *J. of Colloid Inter. Sci.* **96**, 530-547 (1983)].
27. R.A. Toupin, Elastic materials with couple-stresses, *Arch. Rational Mech. Anal.* **11**, 385-414 (1962). [see also: R.A. Toupin, Theories of elasticity with couple stresses, *Arch. Rational Mech. Anal.* **17**, 85-112 (1962)]
28. R.D. Mindlin, Micro-structure in linear elasticity, *Arch. Rational Mech. Anal.* **16**, 51-78 (1964). [see

also: R.D. Mindlin, Second gradient of strain and surface-tension in linear elasticity, *Int. J. Solids Struct.* **1**, 417-438 (1965); R.D. Mindlin and N.N. Eshel, On first strain-gradient theories in linear elasticity, *Int. J. Solids Struct.* **4**, 109-124 (1968)].

29. E.C. Aifantis, Update on a class of gradient theories, *Mech. Mater.* **35**, 259-280 (2003).
30. V. Novozhilov, On a necessary and sufficient condition for brittle strength, *Prik. Mat. Mek.* **33**, 212-222 (1969). [see also: V. Novozhilov, On a necessary and sufficient criterion for brittle strength, *J. Appl. Math. Mechan.* **33**, 201-210 (1969)].
31. N. Pugno and R. Ruoff, Quantized Fracture Mechanics, *Phil. Mag.* **84**, 2829-2845 (2004).
32. J. Ning and E.C. Aifantis, Notes from gradient elasticity boundary value problems, MTU (1997). [see also: G. Efremidis, *Gradient Elasto-plasticity Theory and Size Effects*, PhD Thesis, Aristotle University of Thessaloniki, Greece (2002)].
33. E.C. Aifantis, A proposal for continuum with microstructure, *Mech. Res. Comm.* **5**, 139-145 (1978).
34. E.C. Aifantis, Gradient effects at macro, micro, and nano scales, *J. Mech. Behavior Mats.* **5**, 355-375 (1994).
35. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York (1980).
36. I. Tsagrakis, *The role of gradients in elasticity and plasticity. Analytical and numerical applications*, PhD Thesis, Aristotle University of Thessaloniki, Greece (2001).