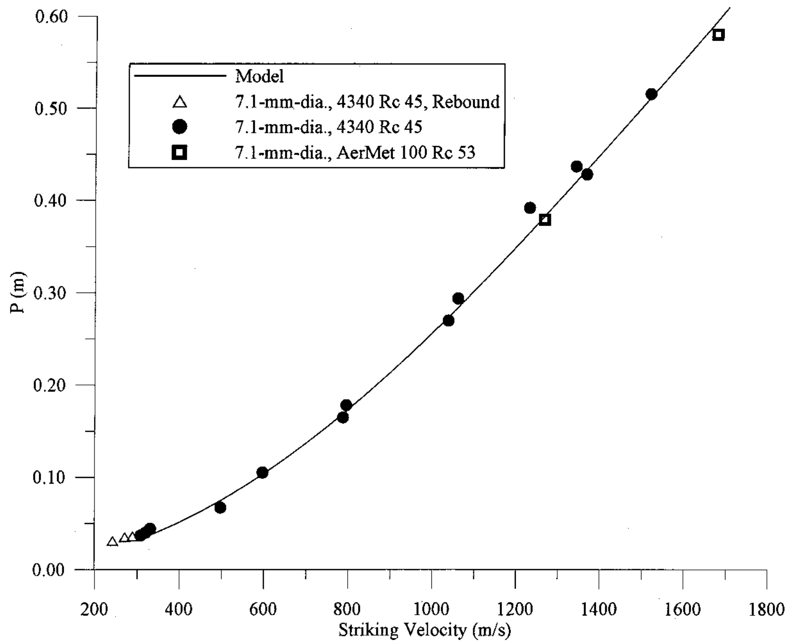

Fig. 1 Data and prediction for limestone targets

less than this limit velocity, the projectiles rebounded from the targets. The penetration limit velocity for these experiments was found to be about 300 m/s.

Acknowledgment

This work was supported by the United States Department of Energy and the Joint DoD/DOE Munitions Technology Development Program. Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AC04-94AL85000.

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One, Two, and Three-Dimensional Universal Laws for Fragmentation due to Impact and Explosion

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Based on the fractal particle size distribution, a fragmentation theory for quasi-brittle materials is herein developed. The results are three simple and powerful universal laws for the multiscale energy dissipation under impact and explosion fragmentation for one, two, and three-dimensional bodies, respectively. The three-dimensional law unifies the most important and well-known fragmentation theories. As an example, it has been applied to the prediction of the devastated area due to asteroid impacts on earth as a function of the energy released in the collision.

[DOI: 10.1115/1.1488937]

1 Introduction

Since the two pioneering books of Mandelbrot [1] and Feder [2], the noneuclidean, fractal, and multiscale geometry of nature has been observed everywhere. In particular, a fractal size distribution is clearly presented by particles obtained from explosive or impact fragmentation processes, both natural and man-made. The fractal nature of the phenomenon simply means that the fragments are geometrically self-similar at each scale. Engleman et al. [3] show that this particle size distribution (power-law) is a necessary consequence of the maximum entropy principle.

Based on the fractal particle size distribution, a fragmentation theory is herein developed. The results are three simple and pow-

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, July 2, 2001; final revision, January 24, 2002. Associate Editor: K. Ravi-Chandar.

erful universal laws for the multiscale energy dissipation under impact and explosion fragmentation for one, two, and three-dimensional bodies, respectively.

The three-dimensional law unifies the most important and well-known fragmentation theories: the surface theory [4], when the dissipation occurs on a surface, the volume theory [5], when the dissipation occurs in a volume and the third comminution theory [6], when the dissipation occurs in a domain exactly intermediate between a surface and a volume (see [7]).

2 Three-Dimensional Theory

After comminution or fragmentation, the cumulative distribution of particles with radius $(=3\sqrt{3/4\pi \cdot \mathrm{volume}_{\mathrm{particle}}})$ smaller than r is (see, for example, [8])

$$P(<r)=\frac{N(<r)}{N_0}=1-\left(\frac{r_{\min}}{r}\right)^D, \tag{1}$$

where $N(<r)$ is the number of fragments with radius smaller than r, N_0 is the total number of fragments, $r_{\min}$ $(\ll r_{\max})$ is the minimum fragment radius, and $D(>0)$ is the fractal dimension.

The probability density function $p(r)$ times the interval amplitude dr represents the percentage of particles with radius comprised between r and $r+\mathrm{d}r$. It is provided by derivation of the cumulative distribution function (1):

$$p(r)=\frac{dP(<r)}{dr}=D\,\frac{r_{\min}^D}{r^{D+1}}. \tag{2}$$

During fragmentation, the energy dissipation due to fracture, dW_F, is proportional to the surface area of fragments, dS (Griffith [9]):

$$\mathrm{d}W_F \propto \mathrm{d}S. \tag{3}$$

During impact fragmentation (material in compression), the main dissipation dW_C is due to collisions and friction between particles (converted into heat) and the effect results to be proportional to the same quantity dS (Smekal [10], see [7]):

$$\mathrm{d}W_C \propto \mathrm{d}S. \tag{4}$$

On the other hand, during explosion fragmentation (material in tension) the main dissipation dW_T is proportional to the kinetic energy of fragmented ejecta dT. The velocity of fragmented ejecta varies inversely with fragment size as $v\propto r^{-1/2}$ (Nakamura and Fujiwara [11]), so that the kinetic energy, i.e., the main dissipation in explosion, results again in being proportional to the fragment surface dS (of volume dV):

$$\mathrm{d}W_T \propto \mathrm{d}T \propto v^2 \mathrm{d}V \alpha \mathrm{d}S. \tag{5}$$

Summarizing, the global dissipation in impacts (W_C+W_F) or explosions (W_T+W_F) surprisingly appears always proportional to the total surface area S of fragments. It can be obtained by integration:

$$\begin{aligned} S&=\int_{r_{\min}}^{r_{\max}} 4\pi r^2 \mathrm{d}N \\ &=\int_{r_{\min}}^{r_{\max}} N_0(4\pi r^2)p(r)\mathrm{d}r \\ &=4\pi N_0\frac{D}{D-2}\,r_{\min}^D\left(\frac{1}{r_{\min}^{D-2}}-\frac{1}{r_{\max}^{D-2}}\right) \\ &\cong \begin{cases} 4\pi N_0\dfrac{D}{D-2}\,r_{\min}^2, & D>2 \\ 4\pi N_0\dfrac{D}{2-D}\,r_{\min}^D r_{\max}^{2-D}, & D<2. \end{cases} \end{aligned} \tag{6}$$

If $0<D<2$ it is necessary to specify $r_{\max}$ but not $r_{\min}$ in order to obtain a finite total surface area of fragments. But if $D>2$ it is necessary to specify $r_{\min}$ in order to constrain the total surface area to a finite value. Thus for most observed distribution of fragments the surface area of the smallest fragments dominates.

On the other hand, the total volume of the particles, or total fragmented volume V, is

$$\begin{aligned} V&=\int_{r_{\min}}^{r_{\max}} \frac{4}{3}\pi r^3 \mathrm{d}N \\ &=\int_{r_{\min}}^{r_{\max}} N_0\left(\frac{4}{3}\pi r^3\right)p(r)\mathrm{d}r \\ &=\frac{4}{3}\pi N_0\frac{D}{3-D}\,r_{\min}^D(r_{\max}^{3-D}-r_{\min}^{3-D}) \\ &\cong \begin{cases} \dfrac{4}{3}\pi N_0\dfrac{D}{3-D}\,r_{\min}^3 r_{\max}^{3-D}, & D<3 \\ \dfrac{4}{3}\pi N_0\dfrac{D}{D-3}\,r_{\min}^3, & D>3. \end{cases} \end{aligned} \tag{7}$$

If $0<D<3$ it is necessary to specify $r_{\max}$ but not $r_{\min}$ in order to obtain a finite volume of fragments. The volume is predominantly in the largest fragments. This is the case for most observed distributions of fragments. If $D>3$ it is necessary to specify $r_{\min}$ but not $r_{\max}$. the volume of the small fragments dominates.

It is interesting to note that in Eqs. (6) and (7) D equal to 2 and 3 do not represent singular points but indeterminate forms. So, the physical meaning is preserved also for D equal to 2 and 3.

Based on fracture mechanics we can assume a material "quantum" of size $r_{\min}$=constant (Novozhilov [12] and Sammis [13]) and make a statistical hypothesis of self-similarity, i.e., $r_{\max} \propto \sqrt[3]{V}$ (the larger the fragmented volume, the larger the largest fragment; Carpinteri [14]), so that the energy W dissipated in a three-dimensional fragmentation process, which is proportional to the total surface area S, can be obtained eliminating N_0 from Eqs. (6) and (7) as

$$W\propto S \propto V^{\bar{D}/3}, \quad \text{with} \begin{cases} \bar{D}=2, & D<2 \\ \bar{D}\equiv D, & 2\leqslant D\leqslant 3 \\ \bar{D}=3, & D>3. \end{cases} \tag{8}$$

The universal law of Eq. (8) can be used to predict the multiscale energy dissipation under fragmentation in impacts and explosions of three-dimensional bodies. It represents an extension of the third comminution theory, where $W\propto V^{2.5/3}$ ([6]; see [7]). The extreme cases contemplated by Eq. (8) are represented by $\bar{D}=2$, surface theory ([4]; see [7]), when the dissipation really occurs on a surface $(W\propto V^{2/3})$, and by $\bar{D}=3$, volume theory ([5]; see [7]), when the dissipation occurs in a volume $(W\propto V)$. These three laws are substantially experimental, so that the universal law of Eq. (8) is obviously experimentally verified.

3 Two-Dimensional Theory

For a two-dimensional body of area A (and thickness h), we have

$$S=\int_{r_{\min}}^{r_{\max}} N(2\pi rh)p(r)dr, \quad A=\int_{r_{\min}}^{r_{\max}} N(\pi r^2)p(r)dr,$$

$$r_{\max}\propto \sqrt[2]{A}, \tag{9}$$

so that Eq. (8) becomes

$$W \propto S \propto A^{\bar{D}/2}, \quad \text{with} \begin{cases} \bar{D}=1, & D<1 \\ \bar{D}\equiv D, & 1 \leqslant D \leqslant 2 \\ \bar{D}=2, & D>2. \end{cases} \tag{10}$$

The universal law of Eq. (10) can be used to predict the multiscale energy dissipation under fragmentation in impacts and explosions of two-dimensional bodies (e.g., panel or shell structures).

4 One-Dimensional Theory

For a one-dimensional body of length L (and cross section h^2), we have

$$S=\int_{r_{\min}}^{r_{\max}} Nh^2 p(r)dr = Nh^2, \quad L=\int_{r_{\min}}^{r_{\max}} Nrp(r)dr = N\bar{r},$$
$$r_{\max} \propto L, \tag{11}$$

so that Eq. (8) becomes ($D>0$)

$$W \propto S \propto L^{\bar{D}}, \quad \text{with} \begin{cases} \bar{D}\equiv D, & D \leqslant 1 \\ \bar{D}=1, & D>1. \end{cases} \tag{12}$$

The universal law of Eq. (12) can be used to predict the multiscale energy dissipation under fragmentation in impacts and explosions of one-dimensional bodies (e.g., beams or cables).

5 An Example of Application: The Asteroid Collision

As an example, we can apply the three-dimensional law to the prediction of the devastated area due to asteroid impacts on earth as a function of the energy released in the collision. The comparison with the experimental Steel's law ([15]), based on nuclear weapons tests, shows a good correspondence.

Assuming that the destroyed zones (or fragmented volumes V) are self-similar at each scale, the area Ω_{devasted} devastated by an impact is proportional to $V^{2/3}$ and, being $W \propto V^{\bar{D}/3}$, the theoretical prediction for the devastated area will be

$$\Omega_{\text{devasted}} \propto W^{2/\bar{D}}. \tag{13}$$

Steel [15] provided the following formula (see http://www1.tpgi.com.au/users/tps-seti/spacegd7.html), based on nuclear weapons tests, for estimating the area of destruction due to asteroid impacts:

$$\Omega_{\text{devasted}} = 400W^{0.67}, \quad [\Omega_{\text{devasted}}]=[\text{km}^2], \quad [W]=[\text{megatons}]. \tag{14}$$

Equation (14) appears in good agreement with the theoretical prediction of Eq. (13) and, if we assume $\bar{D}\approx 3$, they practically coincide.

6 Conclusions

Summarizing, the universal laws for the energy dissipation in impact and explosion fragmentation of one, two, or three-dimensional bodies can be rewritten as

$$\begin{aligned} &W \propto L^{\bar{D}} \quad (0 \leqslant \bar{D} \leqslant 1) \quad \text{one-dimensional} \\ &W \propto A^{\bar{D}/2} \quad (1 \leqslant \bar{D} \leqslant 2) \quad \text{two-dimensional} \\ &W \propto V^{\bar{D}/3} \quad (2 \leqslant \bar{D} \leqslant 3) \quad \text{three-dimensional.} \end{aligned} \tag{15}$$

The three-dimensional law unifies the experimentally verified and well-known fragmentation theories (surface theory, von Rittinger [4]; volume theory, Kick [5]; and third comminution theory, Bond [6]; see [7]).

Acknowledgment

The present research was carried out with the financial support of Ministry of University and Scientific Research (MURST), National Research Council (CNR) and EC-TMR Contract No. ERBFMRXCT960062.

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A Note on the Application of the Flamant Solution of Classical Elasticity to Circular Domains

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It is a well known fact that the Flamant solution of classical elasticity cannot be used at an interior point of an elastic body since the resulting displacement field would be multivalued. In this note we demonstrate that the solution to the problem of a concentrated force at a point on an interior circular boundary has a multivalued displacement component but that the exclusion of the point of application of the load from the domain renders the displacement field single-valued everywhere.

[DOI: 10.1115/1.1480821]

1 Introduction

This note contains an analysis of nonuniform convergence of the displacement field in the Flamant solution to the problem of a concentrated force at a point of an interior circular boundary of an unbounded elastic domain. This issue, which does not exist in the Flamant problem for the straight boundary, arose in previous work by the author on cavity nucleation in planar inclusion problems where the inclusion-matrix interface is modeled explicitly by a

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, July 12, 2001; final revision, March 4, 2002. Associate Editor: M.-J. Pindera.