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Asymptotic analysis of a von Koch beam

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ABSTRACT

Fractal geometry is used in diverse research areas, being an useful tool in describing the mechanical behaviour of natural and man-made structures. In this paper, the structural behaviour of a von Koch cantilever beam is analyzed in the small deformations regime. Analytical recursive formulae for the strain energy scaling are derived, which have been found in good agreement with numerical simulations. Energy considerations suggest a peculiar scaling for the beam rigidity in order to prevent compliance divergence. The results are then extended to evaluate the stiffness matrix of a von Koch beam.

1. Introduction

Fractal geometry was conceived by Mandelbrot in 1975 [1]. Thanks to its great capability in reproducing natural objects, the use of fractal geometry has been successfully implemented in diverse research areas such as mechanics, geology, biology and so on. On the other hand, while the fractal nature of many materials and structures has been widely accepted, e.g. [2,3], the problem of the mechanics of fractal materials and structures is still open. Indeed, the complex geometry of the microstructure makes it impossible to apply the continuum field equations to such objects directly. In other words, fractals are non-differentiable functions and it is not obvious what the derivative of such functions could be. In the last 15–20 years, several approaches have been developed to overcome this drawback.

Size effects on apparent mechanical properties due to the fractal nature of material microstructure have been extensively studied by Carpinteri [4,5] (see also [6]). Carpinteri, by means of the renormalization group (RG) transformations, defined new universal properties (i.e. scale-invariant quantities) having non-conventional or anomalous physical dimensions. Carpinteri and co-workers [7,8], faced the problem of the mechanics of fractal solids successively by means of the local fractional calculus [9]: standard derivatives were replaced with fractional ones associated with the fractal dimension of the deformable domain.

Tarasov [10] proposed to replace the fractal body with a continuum and to describe it by fractional integrals: the fractional integration was then used [11] to determine the dynamics of fractal media.

More recently, Epstein et al. [12] have studied the configuration space of a certain class of deformable fractals by means of the theory of differential spaces of Sikorski. In particular, using the notion of integration with respect to the Hausdorff measure, they have provided a setting for the formulation and numerical solution of problems in the mechanics of such structures.

Another approach to modelling the overall mechanical behaviour of materials with fractal microstructure is to introduce scaling to their properties, for example, the elastic moduli: results were obtained either theoretically, by using the differential self-consistent method [13], or numerically by combining a finite element analysis (FEA) with position-space RG techniques [14]. In order to avoid the computational effort due to the presence of different scales, Soare and Picu [15] have

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recently proposed a finite element procedure with *ad hoc* shape functions which incorporate information about the geometrical complexity.

Eventually, different approaches involve functional analysis. Early attempts were made by Panagiotopolous et al. [16] by means of the theory of Besov spaces. Another important contribution is due to Kigami [17]: an energy form (which recovers the physical meaning of a Laplacian) is constructed on the so-called post-critically finite self-similar fractals as a limit to approximating energies, defined by suitable difference schemes on a sequence of "pre-fractals", e.g. [18,19].

In the present work, the attention is focused onto the von Koch beam [20], due to its great importance in theoretical studies [12,21] and in modelling natural and man-made objects [22] (see also [23] and related references). The paper is organized as follows: the general properties of the (triadic) von Koch beam are briefly recalled (Section 2). The elastic behaviour of a cantilever von Koch beam, in the small deformation regime, is then analyzed under three elementary loading conditions: the couple (Section 3), the transversal force (Section 4), and the longitudinal force (Section 5). Analytical recursive relationships on the strain energy scaling are provided and a method to prevent compliance divergence is proposed. Finally, the stiffness matrix of a von Koch beam is derived on the basis of the results previously obtained (Section 6). Numerical results seem to confirm the validity of the analytical approach.

2. Triadic von Koch beam

Let us recall the properties of the triadic von Koch curve [24,25]. The construction of the von Koch curve starts with a line segment of length l_0, called the *initiator*. At the first iteration, the set consists of four segments of length $l_1 = l_0/3$, obtained by removing the middle third of the initiator and replacing it by the other two sides of the equilateral triangle based on the removed segment. This structure is called the *generator*. The procedure is iterated ad infinitum: at each step the middle third of each interval is replaced by a scaled-down version of the generator (Fig. 1). At the nth step, the number of segments is 4^n with length $l_n = l_0/3^n$; thus, the total length is

$$L_n = 4^n l_n = (4/3)^n l_0. \tag{1}$$

As n tends to infinity, the sequence of the polygonal curves approaches a limiting curve, called the von Koch curve. This is clearly a self-similar set: it is made of four "quarters", each similar to the whole, but scaled-down by a factor 1/3. Its fractal dimension can be determined by exploiting the property of self-similarity, as the ratio of the logarithm of the number of copies to the logarithm of the inverse of the scaling factor. The fractal dimension of the triadic von Koch curve is $D = \ln 4/\ln 3$.

3. Couple

Let us now consider a rectilinear cantilever beam (step 0) subjected to a couple m at the free end. The beam is placed in a Cartesian (x,y) coordinate system in such a way that the left clamped end of the beam is at the origin and the right end, where the couple m is applied, at the point with coordinates $(l_0,0)$ (Fig. 2). The moment M is constant along the beam and equal to m. Focusing our attention to a small deformation regime and assuming a linear elastic isotropic response, the strain energy Φ_0 related to a rectilinear beam could be easily evaluated as

$$\Phi_0 = \frac{1}{2}\int_0^{l_0} \frac{m^2}{EI}\mathrm{d}x = \frac{m^2 l_0}{2EI}, \tag{2}$$

where E is the Young's modulus of the material and I is the moment of inertia of its cross-section with respect to the neutral axis.

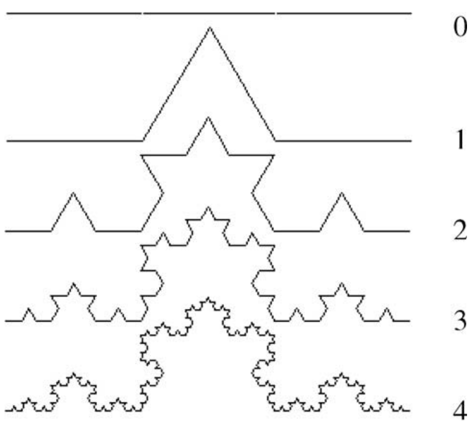

Fig. 1. First four iterations in the von Koch curve generation.

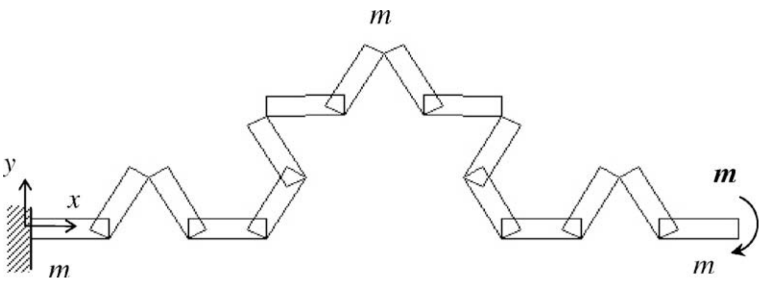

Fig. 2. Diagram of the bending moment for a von Koch cantilever beam (step 2) subjected to a couple m at the free end.

By applying Castigliano's Theorem, it is then possible to calculate the rotation φ_0 at the free end:

$$\varphi_0 = \frac{\partial \Phi_0}{\partial m} = \frac{m l_0}{EI}. \tag{3}$$

On the other hand, if the beam has a self-similar structure, much more can be said about the strain energy and new physical considerations rise up. Henceforth, we will refer to the product $k = EI$ as the beam rigidity. Moreover, the rotation at the free end will be denoted merely by φ_n, where the subscript n refers to the order of iteration.

As already said, the von Koch curve can be seen as the disjoint union of four identical parts, each of which reduced by a factor 3 from the original. In the case of the free-end couple, each part is subjected to the same moment $M = m$ (Fig. 2); hence, it is not difficult to obtain a recursive formula for the strain energy at each step:

$$\Phi_n = \frac{1}{2}\int_S \frac{M^2}{EI}\,\mathrm{d}s = \frac{m^2 L_n}{2k} = \left(\frac{4}{3}\right)^n \Phi_0, \tag{4}$$

where S denotes the structure and Φ_0 is provided by Eq. (2). Eq. (4) shows that, if the rigidity k_0 (implicitly embedded in function Φ_0) remains constant, the strain energy Φ_n increases at each iteration. For n tending to infinity, the structural stiffness tends to zero and the beam becomes infinitely compliant. As deducible from Eq. (3):

$$\varphi_n = \frac{\partial \Phi_n}{\partial m} = \frac{m L_n}{k}, \tag{5a}$$

$$\lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} \left(\frac{4}{3}\right)^n \varphi_0 = \infty. \tag{5b}$$

If the strain energy is supposed to be preserved, the rigidity k must increase as

$$k_n = \left(\frac{4}{3}\right)^n k = \left(\frac{l_n}{l_0}\right)^{1-D} k, \tag{6}$$

where $D = \ln 4 \backslash \ln 3$ is the fractal dimension of the von Koch curve.

This is a simple, yet interesting, result: the rigidity k must increase if the strain energy Φ_n has to be conserved, due to the increased total length L_n. Furthermore, it must scale exactly as $(l_n/l_0)^{1-D}$: in all the other cases, either the strain energy diverges or it converges to zero. The same occurs for the compliance. Assuming the validity of Eq. (6), the rotation at the free end is constant:

$$\varphi_n = \varphi_0 \quad \forall n. \tag{7}$$

4. Transversal force

If the couple m is replaced by a transversal force F, the situation becomes a little more complex. In a generic section of a rectilinear cantilever beam the bending moment varies linearly:

$$M(x) = F(l_0 - x), \tag{8}$$

whereas the shear force is constant and equal to F.

By neglecting the shear and axial compliances (as commonly done in beam-framed structures analysis) the related strain energy Φ_0 is

$$\Phi_0 = \frac{1}{2}\int_0^{l_0} \frac{M^2}{EI}\,\mathrm{d}x = \frac{F^2 l_0^3}{6k}. \tag{9}$$

In this case, Castigliano's Theorem provides the value of the deflection v_0 at the free end:

$$v_0 = \frac{\partial \Phi_0}{\partial F} = \frac{F l_0^3}{3k}. \tag{10}$$

If a von Koch cantilever beam is now considered, it is necessary to evaluate the strain energy related to the next iterations. An analytical recursive expression is not so directly evaluable as in the previous case: the bending moment M is not constant any more, but it varies linearly on each segment constituting the structure (Fig. 3).

In order to find a recursive relationship, the strain energy was computed analytically for the first three iterations and the results were then extended to the next iterations:

$$\begin{aligned}
\Phi_1 &= \left(\tfrac{4}{3} - \tfrac{2}{27} B_0\right) \tfrac{F^2 l_0^3}{6k} = B_1 \Phi_0, \\
\Phi_2 &= \left[\left(\tfrac{4}{3}\right)^2 - \tfrac{2}{27} B_1\right] \tfrac{F^2 l_0^3}{6k} = B_2 \Phi_0, \\
\Phi_3 &= \left[\left(\tfrac{4}{3}\right)^3 - \tfrac{2}{27} B_2\right] \tfrac{F^2 l_0^3}{6k} = B_3 \Phi_0, \\
&\vdots \\
\Phi_n &= \left[\left(\tfrac{4}{3}\right)^n - \tfrac{2}{27} B_{n-1}\right] \Phi_0,
\end{aligned} \tag{11}$$

where $B_0 = 1$ from Eq. (9). The validity of this relationship was checked numerically, by using LUSAS ® code, in terms of displacements (Eq. (10)) up to the 6th iteration: the maximum percentage error was found less than 1‰. Eq. (11) can be rewritten as

$$\Phi_n = \left(\frac{4}{3}\right)^n A_n \Phi_0, \tag{12}$$

where

$$A_n = \sum_{i=0}^{n} (-1)^i \left(\frac{1}{18}\right)^i, \quad n \geqslant 0. \tag{13}$$

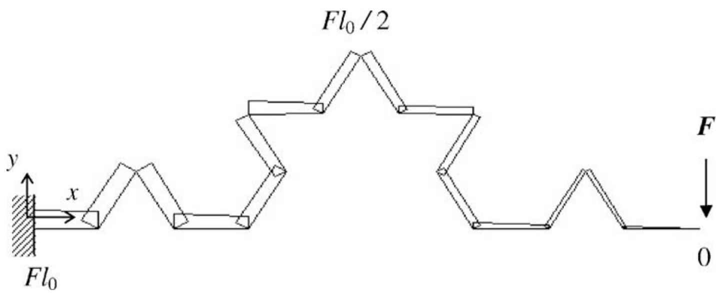

Fig. 3. Diagram of the bending moment for a von Koch cantilever beam (step 2) subjected to a transversal force F at the free end.

Table 1
Multiplying factors A_n related to a von Koch cantilever beam subjected to a transversal force (Eq. (13)) or a longitudinal one (Eq. (21)), respectively

Iteration, n	$A_n(T)$	$A_n(L)$
0	1.0000	NaN
1	0.9444	1.0000
2	0.9475	1.1389
3	0.9474	1.1829
4	0.9474	1.1954
5	0.9474	1.1988
6	0.9474	1.9997
7	0.9474	1.9999
8	0.9474	1.2000
9	0.9474	1.2000
10	0.9474	1.2000
100	0.9474	1.2000
1000	0.9474	1.2000

The values are reported to the first four accurate digits.

The coefficients A_n in Eq. (13) represent a geometric series converging to the value:

$$A = \lim_{n\to\infty} A_n = \sum_{i=0}^{\infty}(-1)^i\left(\frac{1}{18}\right)^i = \frac{18}{19} = 0.947368421. \tag{14}$$

It is interesting to note that, if only the first four accurate digits (which can be considered a good approximation) are taken into account, the multiplying factors A_n already converge after the first three iterations (Table 1):

$$A = A_n \approx 0.9474, \quad n \geqslant 3. \tag{15}$$

The asymptotic result, is thus similar to that obtained in the previous case (i.e. the applied couple): the strain energy Φ_n scales as the total length L_n, unless a multiplying constant. In order to preserve it, the rigidity k must scale as in Eq. (6). Consequently, the deflections at the free end remain finite and different from zero and related to that of the rectilinear cantilever beam ($n = 0$) by the relationship:

$$v_n \approx 0.9474 v_0, \quad n \geqslant 3. \tag{16}$$

5. Longitudinal force

Let us now consider a von Koch cantilever beam subjected to a longitudinal force F (Fig. 4). If a rectilinear beam is considered, the moment M is null in each section of the beam. In order to consider only the bending stiffness, the first order von Koch cantilever beam must then be taken as the "reference iteration". By neglecting again the shear and the axial compliances, the total strain energy of the structure is

$$\Phi_0 = \frac{1}{2}\int_0^{l_0}\frac{M^2}{EI}\mathrm{d}x = \frac{F^2 l_0^3}{108k}, \tag{17}$$

while the horizontal displacement at the free end is provided by Castigliano's Theorem:

$$w_1 = \frac{\partial \Phi_1}{\partial F} = \frac{F l_0^3}{54k}. \tag{18}$$

As done in the previous section, it is possible to demonstrate that, as the order of iteration increases, the strain energy scales as

$$\begin{aligned}
\Phi_2 &= \left[\left(\tfrac{4}{3}\right) + \tfrac{2}{9}B_1 - \tfrac{1}{27}\right]\tfrac{F^2 l_0^3}{108k} = B_2\Phi_1,\\
\Phi_3 &= \left[\left(\tfrac{4}{3}\right)^2 + \tfrac{2}{9}B_2 - \tfrac{1}{81}\right]\tfrac{F^2 l_0^3}{108k} = B_3\Phi_1,\\
&\vdots\\
\Phi_n &= \left[\left(\tfrac{4}{3}\right)^{n-1} + \tfrac{2}{9}B_{n-1} - \tfrac{1}{3^{n+1}}\right]\Phi_1,
\end{aligned} \tag{19}$$

where $B_1 = 1$ from Eq. (17). Eq. (19) can be rewritten in a simpler way:

$$\Phi_n = \left(\frac{4}{3}\right)^{n-1} A_n \Phi_1, \tag{20}$$

where A_n is provided by:

$$A_n = \sum_{i=0}^{n-1}\left(\frac{1}{6}\right)^i - \frac{1}{3^{n+2}}\left(\sum_{i=0}^{n-1}\left(\frac{1}{2}\right)^i - 1\right), \quad n \geqslant 1. \tag{21}$$

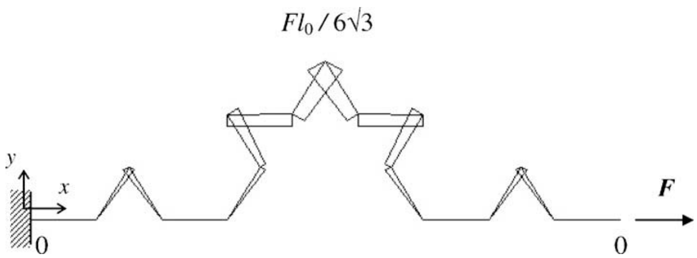

Fig. 4. Diagram of the bending moment for a von Koch cantilever beam (step 2) subjected to a longitudinal force F at the free end.

As n tends to infinity the second terms in Eq. (21) vanishes and the series A_n converges to the value:

$$A = \lim_{n\to\infty} A_n = \sum_{i=0}^{\infty} \left(\frac{1}{6}\right)^i = \frac{6}{5} = 1.2000. \tag{22}$$

In this case, the multiplying factors A_n remain constant (to four accurate digits) after the first eight iterations (Table 1):

$$A = A_n \approx 1.2000, \quad n \geqslant 8. \tag{23}$$

In order to prevent compliance divergence, the beam rigidity k must scale again as in Eq. (6). In this case, the total strain energy tends to that related to the first order von Koch cantilever beam, unless a multiplying constant. Obviously, the same occurs for the horizontal displacements at the free end:

$$w_n \approx 0.9000 w_1, \quad n \geqslant 8. \tag{24}$$

6. Stiffness matrix

In the previous sections a von Koch cantilever beam loaded by the three elementary forces (the couple, the transversal force and the longitudinal force) has been considered. For each case, the strain energy scaling has been found. These laws can be unified referring to the first order von Koch cantilever beam as

$$\Phi_n = \frac{3}{4}\left(\frac{l_n}{l_0}\right)^{1-D} A_n^* \Phi_1, \tag{25}$$

where A_n^* is equal to 1 in the case of the applied couple (Eq. (4)), and coincides with A_n in the cases of transversal force (Eq. (12)) or longitudinal force (Eq. (20)).

By scaling the beam rigidity as in Eq. (6), the total strain energy remain finite for each of the contemplated cases. As a consequence, even the displacements at the free end remain finite and different from zero (see Table 1).

These results could be extended in order to compute the stiffness matrix of a von Koch beam. Let us consider a first order von Koch beam. Its degrees of freedom are numbered in the following order: rotation and y- and x-displacements of the left end, rotation and y- and x-displacements of the right end. The rotations are assumed positive if counter-clockwise. The beam is then clamped at both the ends (Fig. 5). The stiffness matrix coefficients K_{ij} can be obtained by imposing the three unit displacements of the supports and evaluating the corresponding reactions at both the ends (Fig. 6). In order to achieve dimensional homogeneity, it is convenient to multiply the rotational variables by l_0 and to divide the moment variables by the same length. By means of the Principle of Virtual Work, the following (6×6) stiffness matrix is computed:

$$[K]_1 = \frac{k}{l_0^3}\begin{bmatrix} \frac{573}{140} & \frac{81}{14} & -\frac{18\sqrt{3}}{5} & \frac{237}{140} & -\frac{81}{14} & \frac{18\sqrt{3}}{5} \\ \frac{81}{14} & \frac{81}{7} & 0 & \frac{81}{14} & -\frac{81}{7} & 0 \\ -\frac{18\sqrt{3}}{5} & 0 & \frac{432}{5} & \frac{18\sqrt{3}}{5} & 0 & -\frac{432}{5} \\ \frac{237}{140} & \frac{81}{14} & \frac{18\sqrt{3}}{5} & \frac{573}{140} & -\frac{81}{14} & -\frac{18\sqrt{3}}{5} \\ -\frac{81}{14} & -\frac{81}{7} & 0 & -\frac{81}{14} & \frac{81}{7} & 0 \\ \frac{18\sqrt{3}}{5} & 0 & -\frac{432}{5} & -\frac{18\sqrt{3}}{5} & 0 & \frac{432}{5} \end{bmatrix}. \tag{26}$$

Note that because of the symmetry and equilibrium ($K_{41} = K_{21} - K_{11}$ and $K_{12} = K_{22}/2$), only four coefficients have to be determined: K_{11}, K_{22}, K_{33} and K_{31}.

As the iteration process progresses, the stiffness matrix is expected to vary. Anyway, a recursive relationship on each stiffness coefficient is not so direct. Clearly, since the total length diverges, the stiffness vanishes. From the results presented in Table 1 it can be argued that, if the beam rigidity scales as in Eq. (6), after 8 iterations the stiffness factors become substantially constant. In other words, it is possible to write the stiffness matrix of the generic n-order of a von Koch beam as

$$[K]_n = \frac{4}{3}\left(\frac{l_n}{l_0}\right)^{D-1} \frac{k}{l_0^3} [\overline{K}]_n, \tag{27}$$

with

$$[\overline{K}]_n = \begin{bmatrix} C_n & \frac{D_n}{2} & -F_n & \frac{D_n}{2} - C_n & -\frac{D_n}{2} & F_n \\ \frac{D_n}{2} & D_n & 0 & \frac{D_n}{2} & -D_n & 0 \\ -F_n & 0 & E_n & F_n & 0 & -E_n \\ \frac{D_n}{2} - C_n & \frac{D_n}{2} & F_n & C_n & -\frac{D_n}{2} & -F_n \\ -\frac{D_n}{2} & -D_n & 0 & -\frac{D_n}{2} & D_n & 0 \\ F_n & 0 & -E_n & -F_n & 0 & E_n \end{bmatrix}, \tag{28}$$

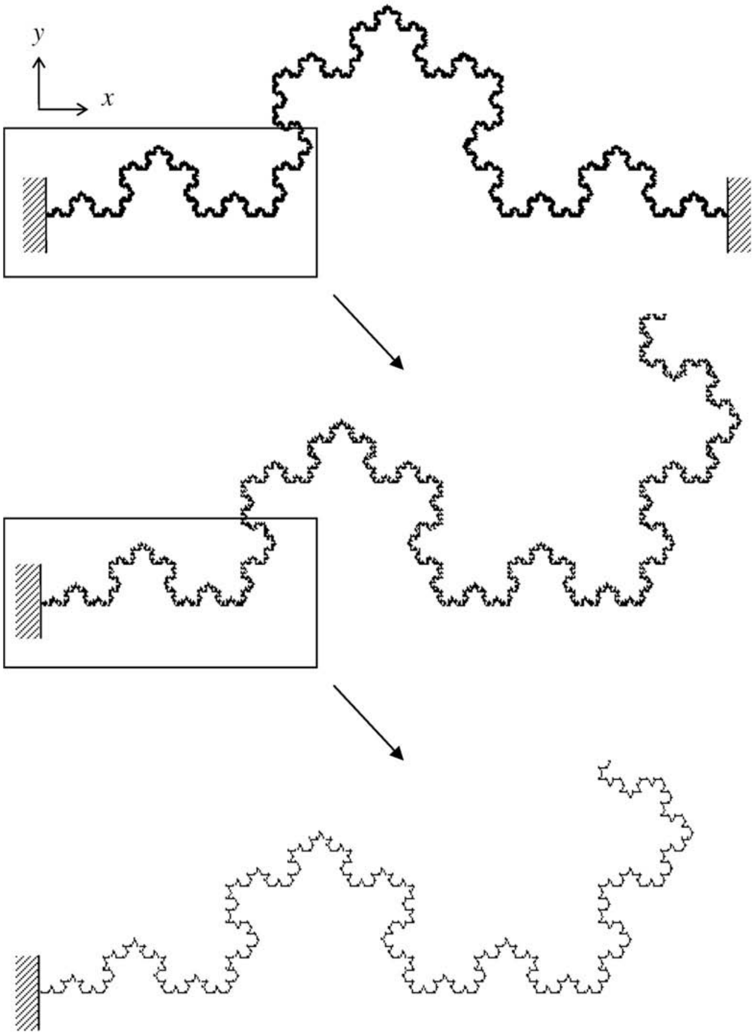

Fig. 5. Self-similarity of a six-order von Koch beam clamped at both the ends.

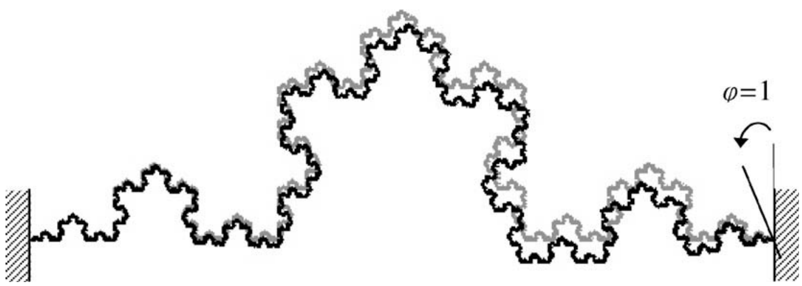

Fig. 6. Elastic deformation of a six-order von Koch beam clamped at both the ends and subjected to a unitary counter-clockwise rotation of the right end.

where the coefficients C_n, D_n, E_n and F_n are supposed to remain invariant after the 8th iteration:

$$[\overline{K}] \approx [\overline{K}]_n, \quad n \geqslant 8. \tag{29}$$

In order to check the validity of such a statement, the four independent stiffness coefficients related to the first six von Koch beam iterations were evaluated numerically by using LUSAS ® code; the results are presented in Table 2. Unfortunately, it was impossible, with our software, to proceed further.

Table 2
Stiffness coefficients C_n, D_n, E_n and F_n related to the first six iterations of the von Koch beam

Iteration, n	1	2	3	4	5	6
C_n	4.0928	4.3927	4.4783	4.4974	4.5018	4.5031
D_n	11.5714	11.3922	11.3021	11.2703	11.2610	11.2598
E_n	86.4000	97.6889	100.6736	101.2127	101.3058	101.3149
F_n	6.2354	8.8089	9.5333	9.6993	9.7379	9.7465

Note that all the coefficients converge to a finite value, the maximum percentage difference between the last two iterations being less than 1‰. Even if two other iterations would lead to more accurate results, the stiffness matrix of the n-order von Koch beam (Eq. (28)) can be written from the stiffness coefficients values reported in Table 2 with a good approximation.

As it is evident from Eq. (26), if the beam rigidity k scales as in Eq. (6), the resulting stiffness matrix $[K']$ remain finite and different from zero:

$$[K'] = \frac{4}{3}\frac{k}{l_0^3}[\overline{K}]. \tag{30}$$

Results could be generalized if a random von Koch beam is considered. For example, each time the middle third of a segment is removed, a coin might be tossed to determine whether to position the new part above or below the removed segment. The asymptotic behaviour is supposed not to change: after a finite number of iterations, the strain energy Φ_n will scale as in Eq. (25), even if some coefficients A_n^* are expected to vary slightly.

Eventually, if a von Koch beam with a different indentation angle θ is analyzed (in this paper it has always been considered $\theta = 60°$), a different Hausdorff dimension must be taken into account according to the relationship [22]:

$$D = \frac{\ln 4}{\ln 2(1+\cos\theta)}. \tag{31}$$

7. Conclusions

In this work, the deformation of a cantilever von Koch beam has been analyzed. The structure has been loaded by the three elementary loadings: couple, transversal or longitudinal force. The recursive scaling of the strain energy has been evaluated analytically and checked numerically. In order to prevent compliance divergence, the scaling of an unique mechanical parameter k, the beam rigidity, has been proposed. The results have then been extended for the evaluation of the stiffness matrix of a von Koch beam, which can be useful in diverse research areas e.g., for optimizing fractal antennas. An analytical recursive expression is presented, which has been found in good agreement with the numerical simulations.

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