Chaos, Solitons and Fractals 41 (2009) 881–885

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

Predominant natural red-shift of quasi-conservative nonlinear systems

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A R T I C L E I N F O

Article history:
Accepted 17 April 2008

A B S T R A C T

Recent discoveries of nonclassical nonlinear phenomena are attracting a large interest in the scientific community, especially in material science. In spite of this, the natural frequency shift related to the appearance of such phenomena remains partially unclear. In this paper, we apply the general and only recently developed Interaction Box Formalism for investigating if a universality in the natural frequency shift of quasi-conservative nonlinear systems exists. Such universality clearly emerges as a rupture in the symmetry, usually leading to a red-shift, quantifiable as a function of the higher- and sub-harmonic generation.

1. Introduction

Nonlinear (NL) nonclassical (NC) phenomena have been recently discovered in material responses, suggesting the existence of a nonlinear mesoscopic elasticity, as emphasized by Guyer and Johnson [1]. Since the field of NLNC elasticity has seen a remarkable progress in recent years [2–7], many of the results achieved within that context could be successfully transferred in different and crucial subjects as the structural [8–10] or human health [11–13] monitoring. Accordingly, the understanding of NLNC effects could represent a strategic target also for different scientific areas, as for example the cancer therapy [12,13].

Material memory leads to the so-called "slow dynamics" [3], when involving slow (i.e., of the order of the day) relaxation or creep phenomena. Faster memory effects (usually arising at the microsecond time-scale) are also present in the so-called "fast dynamics". In addition, the appearances in quasi-static and dynamic material science experiments of (i) higher harmonics, (ii) hysteretic behaviours, and (iii) (resonance) frequency shifts, usually observed downwards, i.e. towards the "red", by increasing the excitation amplitude, reveal "fast dynamics" NL effects. In this context, Hirsekorn and Delsanto [14] have recently developed a fully general Interaction Box Fomalism (IBF); it was applied to prove the existence of a universality in NLNC phenomena, i.e., that the appearance of higher harmonics (i) in general implies hysteretic behaviours (ii). The extension to sub-harmonic generation has recently been presented [15]. Complementarily, in this paper, we apply such a formalism to prove the universality of the fast dynamics, i.e., that the appearance of higher and sub-harmonics (i) implies a frequency shift (iii). A rupture in the symmetry, quantifiable in our treatment, is observed as a common tendency towards a red-shift. Material memory can be included in the approach. Several examples of application and a comparison between our analysis and some experimental observations on damaged or undamaged NLNC material responses conclude the paper.

2. The natural frequency shift of quasi-conservative nonlinear systems

The IBF considers a system as an unspecified "black box" B; the input in the box is the "cause" C whereas the output, after the interaction C–B, is the "effect" E (Fig. 1). The box B can always be separated in its classical linear part $B_L = \omega^2 E$, described

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doi:10.1016/j.chaos.2008.04.016

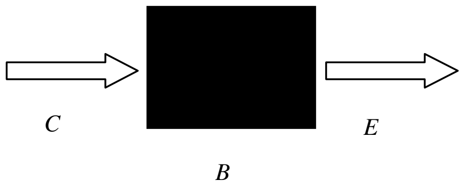

Fig. 1. Interaction box formalism: the cause C interacts with a unspecified black box B, causing the effect E.

by a fundamental circular frequency ω, and in its complementary nonlinear part B_{NL}. The global nonlinear system has a fundamental circular frequency that we denote by ω^*, coincident with ω only for vanishing B_{NL}. B_{NL} is usually assumed to be a function of the effect E and of its time derivative $\dot{E}$. On the other hand, considering in addition a dependence also on its integral $\underset{\cdot}{E} = \frac{1}{\tau}\int_{t-\tau}^{t} E(\xi)\mathrm{d}\xi$, in which τ represents the "memory time", we can in principle model material memory effects: the material has a memory of what happens during the time interval τ. Such a memory time is expected to be of the order of microseconds. For $\tau \to 0$, the memory vanishes and $\underset{\cdot}{E} \to E$. Mathematically the IBF is written as

$$\ddot{E}(t) + \omega^2 E(t) + B_{\mathrm{NL}}(E, \dot{E}, \underset{\cdot}{E}) = C(t) \tag{1}$$

We focus our attention onto steady-state effects E, assumed, according to the trigonometric Fourier series, in the following form:

$$E(t) = E_0 + \sum_{n=1}^{\infty} E_n \cos(n\Omega t + \varphi_n) = E_0 + \sum_{n=1}^{\infty} E_{Cn}\cos(n\Omega t) + E_{Sn}\sin(n\Omega t) = \langle E\rangle + \Delta E(t) \tag{2}$$

where $E_n^2 = E_{Cn}^2 + E_{Sn}^2$ and φ_n are constants and $E_0 = \langle E\rangle$ represents the mean value of $E(t)$ in a period $P = 2\pi/\Omega$, where Ω is the circular frequency of the effect. Similarly to E, let us develop B_{NL} and C in Fourier trigonometric series, as

$$\begin{aligned} B_{\mathrm{NL}}(E(t), \dot{E}(t), \underset{\cdot}{E}(t)) &= B_{\mathrm{NL0}}(E_0, E_{Cn}, E_{Sn}) + \sum_{n=1}^{\infty} B_{\mathrm{NL}Cn}(E_0, E_{Cn}, E_{Sn})\cos(n\Omega t) + B_{\mathrm{NL}Sn}(E_0, E_{Cn}, E_{Sn})\sin(n\Omega t) \\ &= \langle B_{\mathrm{NL}}(E_0, E_{Cn}, E_{Sn})\rangle + \Delta B_{\mathrm{NL}}(E_0, E_{Cn}, E_{Sn}) \end{aligned} \tag{3}$$

$$C(t) = C_0 + \sum_{n=1}^{\infty} C_{Cn}\cos(nm\Omega t) + C_{Sn}\sin(nm\Omega t) = \langle C\rangle + \Delta C \tag{4}$$

where m is a fixed natural number that, if larger than one, allows one to take into account also the appearance of sub-harmonics, i.e., to describe complex phenomena and transition towards deterministic chaos [9,10,15]. Classically we have $m = 1$ [8,14]. The cause C has a period P/m and thus also period P. Note that in the Fourier expansions the coefficients of B_{NL} are nonlinearly related to those of E. If the form of $B_{\mathrm{NL}}(E, \dot{E}, \underset{\cdot}{E})$ is specified, such nonlinear relations are accordingly derivable.

By integrating Eq. (1) over the common period P and assuming for E the form in Eq. (2), we obtain:

$$\omega^2\langle E\rangle + \langle B_{\mathrm{NL}}\rangle = \langle C\rangle \tag{5}$$

Such an equation corresponds to the harmonic balance for the first order term. For higher order terms, the balance may be written as

$$(-n^2\Omega^2 + \omega^2)E_{C,Sn} + B_{\mathrm{NL}C,Sn} = C_{C,Sn/m}\delta_{n/m,\mathrm{int}} \tag{6}$$

where $\delta_{n/m,\mathrm{int}}$ is equal to zero if n/m is not an integer number, or it is equal to one if n/m is an integer number. Harmonics described by $n < m$ are sub-harmonics and only the harmonics described by $n = qm$, q being a positive integer number, are classical higher harmonics.

It is important to note that from Eq. (5) an offset $\langle E\rangle$ in the response is expected if and only if $\langle B_{\mathrm{NL}}\rangle - \langle C\rangle \neq 0$. Consequently, also for $\langle C\rangle = 0$ an offset will appear for $\langle B_{\mathrm{NL}}\rangle \neq 0$.

On the other hand, multiplying Eq. (1) times E and integrating the result over the period P, noting that $\ddot{E}E + \dot{E}^2 = \mathrm{d}/\mathrm{d}t(E\dot{E})$, $\dot{E}E|_0^P = 0$ and thus $\int_0^P \ddot{E}E\,\mathrm{d}t = -\int_0^P \dot{E}^2\,\mathrm{d}t$, yields the following equation:

$$-\int_0^P \dot{E}^2\,\mathrm{d}t + \omega^2\int_0^P \Delta E^2\,\mathrm{d}t + \int_0^P \Delta E\Delta B_{\mathrm{NL}}\,\mathrm{d}t = \int_0^P \Delta E\Delta C\,\mathrm{d}t \tag{7}$$

in which the offsets $\langle E\rangle$, $\langle B_{\mathrm{NL}}\rangle$, $\langle C\rangle$ annul each other.

By virtue of this last equality and of the orthogonality of the trigonometric functions, introducing E as given in Eq. (2) into Eq. (7), we deduce

$$\Omega(K) = \sqrt{\frac{\omega^2\sum_{n=1}^{\infty}E_n^2 + K}{\sum_{n=1}^{\infty}E_n^2 n^2}}, \quad K = 2\langle\Delta E(\Delta B_{\mathrm{NL}} - \Delta C)\rangle \tag{8}$$

i.e., the frequency of the effect (the same result can be obtained by summing Eq. (6) for each value of n).

Imagine to be interested in the resonance frequency Ω', i.e., in the frequency corresponding to the maximum amplitude of the effect when varying the frequency of the cause C; for such a case we have to search the constant K satisfying the following condition:

$$\frac{\partial E(t,\Omega)}{\partial t}=0\rightarrow t=t',\quad E'(\Omega)=E(t',\Omega),\quad \frac{\mathrm{d}E'(\Omega)}{\mathrm{d}\Omega}=0\rightarrow K=K'\rightarrow\Omega'=\Omega(K') \tag{9}$$

The procedure described in Eq. (9) gives K' and thus the resonance frequency as $\Omega'=\Omega(K')$, as well as the related frequency shift $\frac{\Omega'-\omega}{\omega}$; it can be solved numerically. Note that, assuming a monochromatic cause, i.e., $C(t)=C\cos(\Omega t)$ and a weak nonlinearity we deduce $t'\approx\frac{1}{\omega}\frac{\sum_{n=1}^{\infty}nE_{Sn}}{\sum_{n=1}^{\infty}n^2E_{Cn}^2}$. On the other hand, K' cannot be found analytically, even if we expect, as a first approximation, $\Omega'\approx\omega^*$ (the resonance will take place around the natural frequency) and thus $K'\approx K^*=K(\Delta C\rightarrow 0)=2\langle\Delta E\Delta B_{\mathrm{NL}}\rangle$.

Since we are interested in the natural frequency ω^* of nonlinear systems, we have to consider a vanishing cause. Assuming a vanishing cause in Eq. (8) implicitly implies the assumption of a quasi-conservative system (i.e., a system that can still exchange energy but with a null mean value over a period), to justify the validity of Eq. (2) also when the cause C vanishes. Accordingly, the natural frequency shift is predicted as

$$\frac{\Delta\omega}{\omega}=\frac{\omega^*-\omega}{\omega}=\sqrt{\frac{\sum_{n=1}^{\infty}E_n^2+K^*\omega^{-2}}{\sum_{n=1}^{\infty}E_n^2n^2}}-1,\quad K^*=K(\Delta C\rightarrow 0)=2\langle\Delta E\Delta B_{\mathrm{NL}}\rangle \tag{10}$$

and the role of the high- and sub-harmonics on it is evident and quantified.

For weak nonlinearities, i.e. for $B_{\mathrm{NL}}\ll\omega^2E$, we can consider $m=1$ and Eq. (10) becomes

$$\frac{\Delta\omega}{\omega}=\frac{\omega^*-\omega}{\omega}\approx\frac{1}{2}\frac{\sum_{n=1}^{\infty}(1-n^2)E_n^2+K^*\omega^{-2}}{\sum_{n=1}^{\infty}E_n^2n^2}\approx\frac{-3E_2^2-8E_3^2+K^*\omega^{-2}}{2A^2} \tag{11}$$

in which we have assumed $A\equiv E_1\gg E_n^2\gg E_{n+2}^2\ \forall n>1$, where A is the amplitude of the effect.

3. Classification of the nonlinear systems

The generality of Eq. (10) allows one to classify different nonlinear systems by varying the parameter K^*. In particular a system displaying a red-shift must have

$$K^*<K_+^*=\omega^2\sum_{n=1}^{\infty}E_n^2(n^2-1)>0 \tag{12}$$

or conversely will display a blue-shift if

$$K^*>K_+^*=\omega^2\sum_{n=1}^{\infty}E_n^2(n^2-1)>0 \tag{13}$$

A "hidden" system described by

$$K^*=0 \tag{14}$$

for which thus the weak box information survived in Eq. (10) fully disappears, will surely display a red-shift. For example, the nonlinearity described by $B_{\mathrm{NL}}=g(E)\dot{E}$, where g is a unspecified function (that has to satisfy the hypothesis of a steady state condition for E, thus of a quasi-conservative system) corresponds to a hidden system. In fact, in general $K^*=\frac{2}{P}\int_0^P(E-\langle E\rangle)(B_{\mathrm{NL}}-\langle B_{\mathrm{NL}}\rangle)=2\langle EB_{\mathrm{NL}}\rangle+2\langle E\rangle\langle B_{\mathrm{NL}}\rangle$, whereas in our case $\langle B_{\mathrm{NL}}\rangle=\frac{1}{P}\int_0^P g(E)\dot{E}\,\mathrm{d}t=\frac{1}{P}\int_0^P g(E)\,\mathrm{d}E=0$ since $E|_0^P=0$, and similarly $\langle EB_{\mathrm{NL}}\rangle=0$.

In addition, real systems must have

$$K^*>K_-^*=-\omega^2\sum_{n=1}^{\infty}E_n^2<0 \tag{15}$$

as can be evinced imposing the reality of the square root in Eq. (10).

4. Example of application

Consider the hidden system described by $B_{\mathrm{NL}}=\gamma E\dot{E}\ll\omega^2E$. In this case $g(E)=\gamma E$. By virtue of Eq. (6), we find $E_0=0$, $E_{C2}=\frac{\gamma E_{C1}E_{S1}}{3\omega}$, $E_{S2}=\frac{\gamma(E_{S1}^2-E_{C1}^2)}{6\omega}$, thus $E_2=\frac{\gamma A^2}{6\omega}$ ($A\equiv E_1\equiv\sqrt{E_{C1}^2+E_{S1}^2}$), and $E_n\approx 0$, $\forall n>3$. By applying Eq. (11), we immediately deduce $\frac{\Delta\omega}{\omega}\approx-\frac{\gamma^2A^2}{24\omega^2}$ (the same result is obtained by assuming *a priori* $E_{C1}=0$ or $E_{S1}=0$, as a choice for time origin).

Now, let us focus on NLNC materials. According to [16], the dynamic behaviour of solids can be modelled assuming a nonlinear and hysteretic elastic modulus M of the material (defined by $M\equiv\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}$ with σ stress and ε strain) in the form of

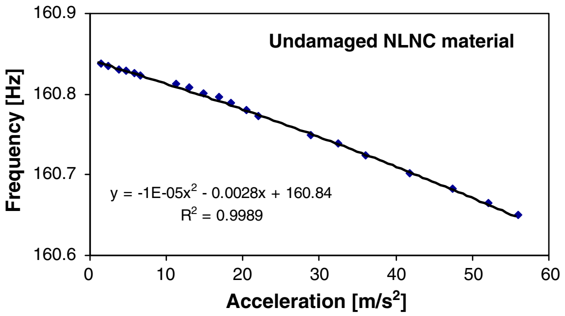

Fig. 2. Frequency vs. acceleration amplitude for undamaged NLNC material. Comparison between experimental observations [17] and an application of our general treatment.

$M \approx M_0(1 - \beta\varepsilon - \delta\varepsilon^2 - \alpha[\Delta\varepsilon + \varepsilon\,\text{sign}(\dot{\varepsilon})])$, where $M_0, \alpha, \beta, \delta$ are constants and $\Delta\varepsilon$ represents the local strain amplitude over the previous period. The last term can be considered as a nonlinear hysteretic damping and it is expected to cause a blue-shift, proportional to $(\alpha\Delta\varepsilon)^2$ [17]. Since the observations on material response emphasizes red-shifts, we focus on the other terms. According to the IBF, the corresponding nonlinear box will be given by $B_{NL} \approx -(\alpha^* AE + \beta^* E^2 + \delta^* E^3)$, where $\alpha^*, \beta^*, \delta^*$ are constants. To give an idea of the effect of each term on the frequency shift, we can apply Eq. (11). It is clear that, if $\alpha^* A \ll \omega^2$, the first term would correspond to a frequency shift of $\frac{\Delta\omega}{\omega} \approx -\frac{\alpha^* A}{2\omega^2}$. Then, we consider the second term, i.e., $B_{NL} = -\beta^* E^2 \ll \omega^2 E$. Since $\langle B_{NL} \rangle \neq 0$, an offset $\langle E \rangle$ in the response is expected according to our analysis. We can assume $\varphi_n = 0$ and search a solution of first approximation in β^*. For such a case, $K^* = -\frac{2\beta^*}{P}\int_0^P (E - \langle E \rangle)E^2\,dt \approx -\frac{2\beta^* A^3}{P}\int_0^P \cos^3(\Omega t)\,dt = 0$, thus this is a hidden system. Correspondingly, from the previous analysis we expect a red-shift, e.g., independently of the sign of β^*. By virtue of Eq. (6), we find $E_0 \approx \frac{\beta^* A^2}{2\omega^2}$, $E_2 \approx -\frac{\beta^* A^2}{6\omega^2}$ and $E_n \approx 0, \forall n > 2$. Thus, by applying Eq. (11), we immediately derive $\frac{\Delta\omega}{\omega} \approx -\frac{\beta^{*2} A^2}{24\omega^4}$. Finally, we treat the third term $B_{NL} = -\delta^* E^3 \ll \omega^2 E$. Since $\langle B_{NL} \rangle \approx 0$, $\langle E \rangle \approx 0$. In addition $K^* \approx -\frac{2\delta^*}{P}\int_0^P E^4\,dt \approx -\frac{2\delta^* A^4}{P}\int_0^P \cos^4(\Omega t)dt = -\frac{3A^4\delta^*}{4}$ $(\varphi_n = 0)$. By virtue of Eq. (6), we find $(E_0 = 0)$ $E_2 = 0$, $E_3 = \frac{\delta^* A^3}{32\omega^2}$ and $E_n \approx 0, \forall n > 3$. By applying Eq. (11) (in which the contribute of E_3 becomes negligible with respect to that of K^*), we immediately deduce $\frac{\Delta\omega}{\omega} \approx -\frac{3\delta^* A^2}{8\omega^2}$. Since the frequency shift related to β^* is formally included in that of δ^* (note that the vice versa is not true), for the sake of simplicity we assume $\beta^* \approx 0$. As a matter of fact, it is clear that the frequency shifts related to α^* and δ^* do not interact (weak nonlinearities): thus, if both are present simultaneously, the frequency is predicted as $\omega^* \approx \omega - \frac{\alpha^*}{2\omega}A - \frac{3\delta^*}{8\omega}A^2$.

We refer now to the observations on frequency shift versus acceleration amplitude for damaged or undamaged NLNC materials [17]. Assuming for the sake of simplicity a NLNC material as previously treated, we expect a dependence of the

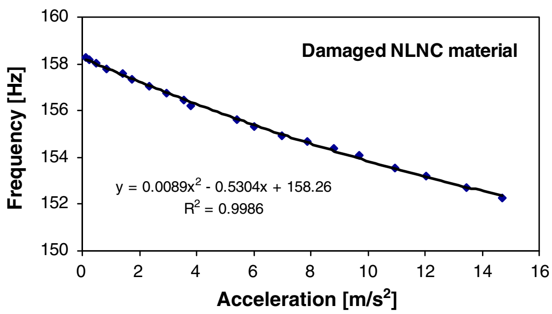

Fig. 3. Frequency vs. acceleration amplitude for damaged NLNC material. Comparison between experimental observations [17] and an application of our general treatment.

Table 1
Black box B parameters identified from the observation of the effect E on NLNC undamaged or damaged materials [17]: note their strong modification imposed by the damage ([SI] units)

Black box parameters	f_0	α^*	δ^*
Undamaged (1)	160.84	36×10^6	17.7×10^{10}
Damaged (2)	158.26	66×10^8	-14.5×10^{13}
Ratio (2/1)	0.98	180	−800

resonance frequency $f = \Omega^*/(2\pi) \approx \omega^*/(2\pi)(f_0 = \omega/(2\pi))$ in the form of $f(X) \approx f_0 - \frac{\alpha^*}{32\pi^4 f_0^3}X - \frac{3\delta^*}{512\pi^6 f_0^5}X^2$, where $X \approx \omega^2 A$ is the amplitude of the acceleration. The comparisons between our predictions and the experimental observations are shown in Fig. 2 for undamaged or in Fig. 3 for damaged NLNC material. The "black box" B parameters f_0, α^*, δ^* are correspondingly identified by observation of the effect E, as reported in Table 1. The damage has strongly modified them: the zero order component, i.e., the fundamental frequency, has been slightly reduced by the presence of damage (2%); the first order component is changed by two order of magnitude, whereas the second order component is changed by three orders of magnitude and, more interestingly, also its sign has changed (see Table 1). Thus, the variation in the first or second order effect, rather then in the zero order one, could represent an important tool for material damage monitoring. If the change of the sign of the second order component will be confirmed by future experimental investigations, the sign itself could represent an interesting On/Off tool for industrial quality production control.

5. Conclusions

Summarizing, our analysis clearly demonstrates that a universal asymmetry in the natural frequency shift is expected and quantifiable for quasi-conservative nonlinear systems: red-shifts are more likely and seem to be due to the appearances of high- and sub-harmonics, as quantified by Eqs. (10) or (11). Only systems with sufficiently large values of K^* could escape from a red-shift (Eq. (13)). The applications here reported are just examples of the proposed general treatment, evidently applicable in different scientific areas.

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