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LETTERS IN FRACTURE AND MICROMECHANICS

ON LINEAR ELASTIC FRAGMENTATION MECHANICS UNDER HYDROSTATIC COMPRESSION

Nicola M. Pugno and Alberto Carpinteri
Department of Structural Engineering and Geotechnics, Politecnico di Torino,
Corso Duca degli Abruzzi 24, 10129 Torino, Italy.
Tel: +39.115644895; Fax: +39.115644899; E-mail: nicola.pugno@polito.it
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Abstract. In this letter we apply the Griffith's energy balance of Linear Elastic Fracture Mechanics to predict the hydrostatic compressive strength of granular matter, as a first step towards a Linear Elastic Fragmentation Mechanics. The result is a simple law for the critical pressure, in strict analogy to the Griffith's strength prediction, found to be mainly a function of the surface to volume ratio of particles. The influence of compaction and fragmentation evolution on the compressive strength is also quantified. Thus, this letter presents one possible approach to fragmentation mechanics.

Keywords: fragmentation, comminution, hydrostatic compression, fractal, crack

1. Introduction. Granular materials are very versatile systems, behaving as fluid if unconfined or as solid if confined; this versatility can help the today material scientist in designing smart artificial systems and their study in understanding natural, e.g. geological, complex processes. But controlling fragmentation and particle production remains a complex task.

After the pioneer approach to quasi-static fracture introduced by Griffith (1921), by balancing the stored elastic energy against the energy spent in surface creation, Grady (1982) presented a modified treatment for dynamic fragmentation, substituting in the energy balance the stored elastic energy with the kinetic energy. Considering both kinetic and stored elastic energies, Glenn and Chudnovsky (1986) proposed a unification, then refined with Gommerstadt (Glenn, Chudnovsky and Gommerstadt, 1986), resembling the Griffith condition for brittle fracture at low strain rate and the Grady (1982) solution at high strain rate. Pugno (2006) has recently proposed an extension of the energy balance in dynamic fracture, by considering the discrete nature in space and time of the crack propagation. All these approaches are deterministic, and statistical analyses have been also proposed. In particular, energy consumption can be quantified during fragmentation, according to the classical von Rittinger (1876), Kick (1883) or Bond (1952) laws, or their modern statistical unification by Carpinteri and Pugno (2002a,b). In spite of this, only recently a more detailed deterministic/statistical quasi-staic breakage mechanics has been developed by Einav (2007a,b,c). Without competing with this more sophisticated approach, we present here a simple, still quasi-static, treatment towards the development of a Linear Elastic Fragmentation Mechanics, in strict analogy with the pioneer paper by Griffith (1921) on Linear Elastic Fracture Mechanics. Thus, this letter presents one possible approach to fragmentation mechanics.

2. Linear Elastic Fragmentation Mechanics. Consider a granular matter under hydrostatic pressure, Figure 1 (no crack can be envisioned here, just particles).

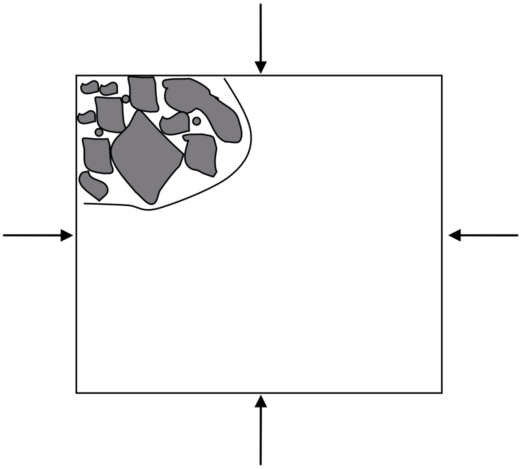

Figure 1. Granular matter under hydrostatic pressure.

The energy balance during crack propagation imposes that the variation of the total potential energy $\mathrm{d}W$, equal to the variation of the elastic strain energy $\mathrm{d}\Phi$ minus the variation of the external work $\mathrm{d}L$, must be equal to the opposite of the work spent in creating the new surface of fragments $\mathrm{d}S$, i.e. $\mathrm{d}W = \mathrm{d}\Phi - \mathrm{d}L = -\gamma \mathrm{d}S$, where γ is the energy spent per unit area (in fragmentation, e.g. by fracture and friction). Assuming linearity for the constitutive law implies the validity of the Clapeyron's theorem, i.e. $\mathrm{d}\Phi = \mathrm{d}L/2$. Accordingly $\mathrm{d}W = -\mathrm{d}\Phi$ and the condition for fragmentation becomes $\mathrm{d}\Phi = \gamma \mathrm{d}S$. In the hypothesis of linearity $\Phi = \frac{p^2 V_n}{2B_n}$, where p is the applied hydrostatic pressure, V_n is the nominal compressed volume and B_n is the nominal bulk modulus. We have to introduce nominal quantities, since the fragment compaction is not complete and the external pressure works on the nominal volume. By integration we find the critical pressure as $p_C = \sqrt{\frac{2B_n \gamma (S - S_0)}{V_n}}$, where S_0 is the initial surface area of particles. Indicating with $c = \frac{V}{V_n}$ the compaction index, in which V denotes the real volume of particles, the compatibility of the displacements suggests $B_n \approx cB$, where B is the intrinsic material bulk modulus. Accordingly:

$$p_C = c\sqrt{2\gamma B\frac{\Delta S}{V}} \tag{1}$$

where $\Delta S = S - S_0$. Eq. (1) shows that an increment in the critical pressure corresponds to new surface creation, i.e. fragmentation, and/or larger compaction of the particles. As a first approximation, the compaction index can be considered as a constant during the loading history, since for identical particles it is size-independent (e.g. related to the Bravais' lattices). Note that eq. (1) coincides with the classical Griffith's formula for $c \to 1$, $B \to E$ (the Young's modulus) and $\frac{\Delta S}{V} = \frac{1}{\pi a}$, where a is the crack half-length. Thus the analogy is evident and the role of the crack length is played by the surface to volume ratio of particles. Note that in the very interesting ref. 12 a similar analogy between the Griffith's strength and the critical comminution pressure has been presented, but with the role of the crack length played only by the particle surface. This approach is complementary to those previously mentioned on dynamic fracture; in fact, even if high strain rates are not considered here, a particle size distribution was not contemplated there (Grady, 1982; Glenn and Chudnovsky, 1986; Glenn, Chudnovsky and Gommerstadt, 1986).

The surface to volume ratio can be simply derived for a given particle size distribution. Often Nature produces, at least in critical conditions, a self-similar (fractal) distribution (Carpinteri and Pugno, 2002a,b). Thus, we consider here such a distribution just as a plausible example of application, even if other distributions can be straightforwardly treated by our approach. The simplest way to characterize the fragment evolution is considering a reduction of the largest fragment, still maintaining the same particle size distribution (that could vary itself (Einav, 2007a,b,c); in particular in these papers the fractal distribution was considered as the limiting distribution, for the finer particles) and the same size of the smallest particle (considered as a material constant). Assuming a fractal size distribution, in which a number of particles $N \propto r^{-D}$ has size smaller than r, where D denotes the fractal dimension of the particle set (often comprised between 2 and 3, i.e. between that of a surface and that of a volume), we obtain:

$$\frac{S}{V} = \frac{\chi}{r_{\min}}\frac{3-D}{D-2}\frac{1-f^{D-2}}{f^{D-3}-1}, \quad f = \frac{r_{\min}}{r_{\max}} \tag{2}$$

where χ is a shape factor, equal to 3 for spherical particles, $r_{\min,\max}$ are the sizes of the smallest and largest particles and thus f can be defined as a "fragmentation index". Note that eq. (2) is well-defined also for D=2,3, for which logarithmic dependencies would appear. Introducing eq. (2), also evaluated for $f_0 = \frac{r_{\min}}{r_{\max 0}}$ ($r_{\max 0}$ is the initial value of the largest fragment), into eq. (1), we predict the critical pressure as a function of the fragmentation evolution f, see Figure 2. Since $\mathrm{d}p_C/df > 0$, the comminution is found to be everywhere stable. In particular, for $f \to 0$ ($2 < D < 3$) $\frac{S}{V} \to \frac{\chi}{r_{\min}} \frac{3-D}{D-2} f^{3-D}$ and considering $f_0 \approx 0$ implies $p_C \to 0$; whereas, for $f = 1-x$ with $x \to 0$ we find $\frac{S}{V} \to \frac{\chi}{r_{\min}}$ and consequently ($f_0 \approx 0$) $p_C \to c\sqrt{\frac{2\gamma B\chi}{r_{\min}}}$. This demonstrates that $p_C \to \infty$ for $r_{\min} \to 0$, so that nanoparticles could support enormous pressures, in agreement with the classical argument by Kendall (1978) and, at a larger size scale, with further experimental observations (Nakata et al., 2001).

The energy spent during comminution is:

$$L = 2\Phi = \gamma \Delta S \tag{3}$$

And, for $S_0 = 0$, $f << 1$ (i.e. $r_{\min} << r_{\max}$) and $r_{\max} \propto \sqrt[3]{V}$ (statistically, the larger the fragmented volume the larger the largest fragment), the "universal" fractal scaling (Carpinteri and Pugno, 2002a,b) $L \propto V^{D/3}$ is recovered.

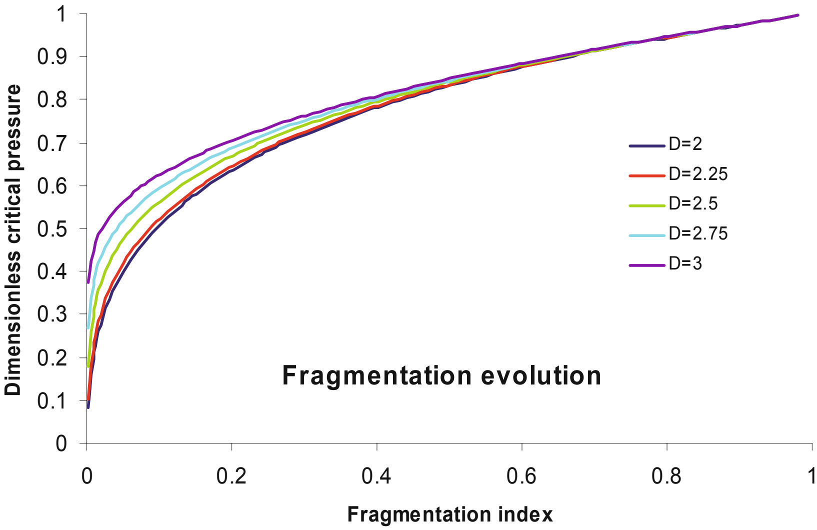

Figure 2. Dimensionless critical pressure $p_C^* = \frac{p_C}{p_{C-\max}} \equiv \frac{p_C}{c}\sqrt{\frac{r_{\min}}{2\gamma B\chi}}$ as a function of the fragmentation index *f*, by varying the fractal dimension *D* ($S_0 = 0$). Note the sharp transition from first fragmentation to subsequent finer comminution (and compaction).

3. Conclusions. Summarizing, we have presented a preliminary step towards a Linear Elastic Fragmentation Mechanics, which suggests the use of nanoparticle-based systems for supporting severe hydrostatic pressures.

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